

packing common bases:

- eg. ① bipartite matching (König)
 ② arborescence packing (Edmonds)

non-example: 

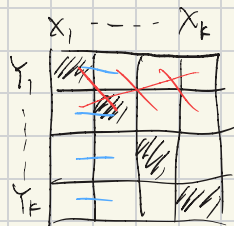
Strongly base-orderable matroids (Def):

Thm: (Davies - McDiarmid): $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$, SBO.

if E can be partitioned into k bases of M_1 $\forall i \in \{1, 2\}$,
 then E can be partitioned into k common bases of M_1 & M_2 .

pf: pick a partition $X_1, \dots, X_k \in \mathcal{B}(M_1)$
 $Y_1, \dots, Y_k \in \mathcal{B}(M_2)$

maximizing $\sum_{i=1}^k |X_i \cap Y_i|$



Assume P.S.O.C., $\exists |X_i \cap Y_j| \neq \emptyset$, $i \neq j$

pick $\Phi_1: X_i \rightarrow X_j$, $\Phi_2: Y_i \rightarrow Y_j$.

$\Rightarrow$ bipartite graph on $X_i \cup X_j \cup Y_i \cup Y_j$ with bipartition $S \cup T$

$$X'_i = (X_i \cup X_j) \cap S, \quad X'_j = (X_i \cup X_j) \cap T.$$

$$Y'_i = (Y_i \cup Y_j) \cap S, \quad Y'_j = (Y_i \cup Y_j) \cap T$$

$$\Rightarrow X'_i \cap Y'_i = S \cap (X_i \cup X_j) \cap (Y_i \cup Y_j)$$

$$X'_j \cap Y'_j = T \cap (X_i \cup X_j) \cap (Y_i \cup Y_j)$$

$$\Rightarrow (X'_i \cap Y'_i) \cup (X'_j \cap Y'_j) = (X_i \cup X_j) \cap (Y_i \cup Y_j) \neq (X_i \cap Y_i) \cup (X_j \cap Y_j)$$

Thm: (Edmonds) arborescence packing.

Submodular flow:

Def. (crossing family)

Def. (crossing submodular)

Def. Let $D=(V,A)$ be a digraph, and $f: E \rightarrow \mathbb{R}$ be crossing submod.
where $\mathcal{C} \subseteq 2^V$ is a crossing family. A submodular flow is

$$\mathcal{F} := \{x \in \mathbb{R}^A : \chi(S^+(v)) - \chi(S^-(v)) \leq f(v) \quad \forall v \in V\}$$

Thm: The submodular flow is box-TDI

Pf. $\max w^T x$

$$\text{s.t.} \quad \chi(S^+(v)) - \chi(S^-(v)) \leq f(v) \quad \forall v \in V$$

$$a_e \leq x_e \leq b_e \quad \forall e \in A$$

$$\min \sum_{v \in V} y_v \cdot f(v) + \sum_{e \in A} b_e \beta_e - \sum_{e \in A} a_e \alpha_e$$

$$\text{s.t.} \quad \sum_{v: e \in S^+(v)} y_v - \sum_{v: e \in S^-(v)} y_v + \beta_e - \alpha_e = w_e \quad \forall e \in A$$

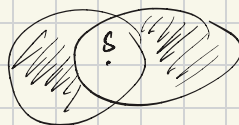
$$y, \alpha, \beta \geq 0$$

Choose optimal y^* minimizing $\sum_{v \in V} y_v^* |U^+ \cap V^-|$.

claim: The support $\{v: y_v^* > 0\}$ is cross-free

claim: Constraint matrix restricted to variables in the support is TU

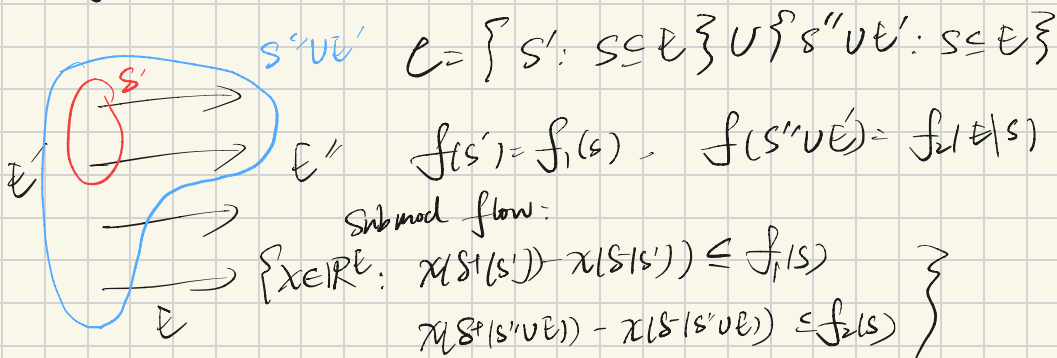
$$\begin{array}{ccc} y & \alpha & \beta \\ \hline E & -I & I \end{array}$$



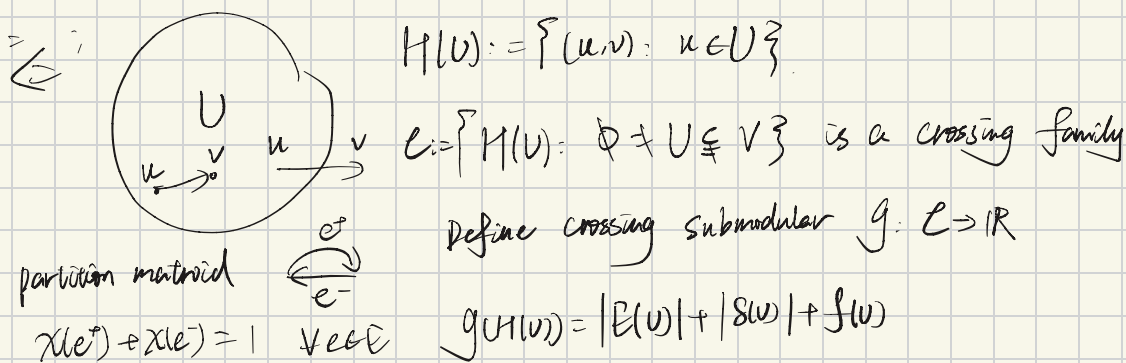
$\Rightarrow$ laminar.

Equivalence of submodular flow & polymatroid intersection

" $\Rightarrow$ " $f_i: 2^E \rightarrow \mathbb{R}_{\geq 0}$ - polymatroids



$$\Leftrightarrow \begin{aligned} \chi(S) &\leq f_1(S) \\ \chi(E|S) &\leq f_2(E|S) \end{aligned} \quad \checkmark$$



$$\chi(H(U)) \leq g(H(U)) \Leftrightarrow \chi(S_A^+(u)) + (|S(u)| - \chi(S_A^-(u))) \leq f(u) + |S(u)|$$

Thm (Luchessi-Torngren)

Thm (Lehman)

Conj (Woodall):

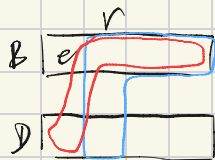
Thm: $\frac{1}{6}$ - approx.

Hardness of packing common bases

Def (sparse paving matroid): Let $H = (E, \mathcal{E})$ be an r -uniform hypergraph, s.t.

Let $P_1, P_2 \in \mathcal{E}$ satisfies $|P_1 \cap P_2| = r-2$. Let $B = \binom{E}{r} \setminus \mathcal{E}$. Then,

$\mathcal{B}$ forms bases of a matroid



Instance: A

B

P_1 --- P_r

r odd
 $|E| = 2r$

 M_1 partition method

M_2 : sparse paving, hypergraph $(E, \mathcal{E})$, where $\mathcal{E} := \{F: |F \cap P_0| = 1 \ \forall i \in [r]; |F \cap A| \text{ odd}\}$

2 disjoint common bases of $M_1 \cap M_2$

$\forall F \in \mathcal{E}$, define M_F : sparse paving - hypergraph $(E, \mathcal{E}(F))$

$\exists$ 2 disjoint common bases of $M_1 \cap M_2^H$.