

Monotone submodular maximization

Cardinality constraints

$f: 2^n \rightarrow \mathbb{R}_{\geq 0}$ monotone, submodular, normalized ($f(\emptyset) = 0$)

$\max \{f(S) : S \subseteq [n], |S| \leq k\}$ eg. max k-cover

Thm: $(1 - \frac{1}{e})$ -approx

Greedy: $A \leftarrow \emptyset$

pick $a \in \arg\max f(a|A)$

$A \leftarrow A \cup \{a\}$

Pf: Assume $Alg = \{a_1, \dots, a_k\}$

$A_i = \{a_1, \dots, a_i\}$

$$f(a_i | A_{i-1}) \geq \frac{1}{k} \sum_{o \in OPT} f(o | A_{i-1}) \stackrel{\text{(submod)}}{\geq} \frac{1}{k} f(OPT | A_{i-1}) \stackrel{\text{(monotone)}}{\geq} \frac{1}{k} (f(OPT) - f(A_{i-1}))$$

$$\Rightarrow f(A_i) - f(A_{i-1}) \geq \frac{1}{k} (f(OPT) - f(A_{i-1}))$$

(continuous) $y_i = f(A_i)$, $T = [0, 1]$, $dt = \frac{1}{k}$

$$\frac{dy}{dt} \geq (f(OPT) - y)$$

$$\Rightarrow \int_0^{f(Alg)} \frac{1}{f(OPT) - y} dy \geq \int_0^1 dt$$

$$\Rightarrow -\ln(f(OPT) - f(Alg)) + \ln(f(OPT)) \geq 1$$

$$\Rightarrow \frac{f(OPT)}{f(OPT) - f(Alg)} \geq e$$

$$f(Alg) \geq (1 - \frac{1}{e}) \cdot f(OPT)$$

(discrete) $f(\text{OPT}) - f(A_i) \leq \alpha (f(\text{OPT}) - f(A_{i-1}))$

$$(1-\alpha) \cdot f(\text{OPT}) \leq f(A_i) - \alpha \cdot f(A_{i-1})$$

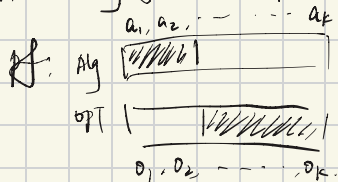
$$\alpha \triangleq 1 - \frac{1}{e}$$

$$\Rightarrow f(\text{OPT}) - f(\text{Alg}) \leq (1 - \frac{1}{e})^k \cdot f(\text{OPT}) \leq \frac{1}{e} \cdot f(\text{OPT})$$

$$\Rightarrow f(\text{Alg}) \geq (1 - \frac{1}{e}) \cdot f(\text{OPT})$$

Matroid Constraints

Thm: Greedy gets $\frac{1}{e}$ -approx.



OK: Alg picks a basis (monotonicity is important)

$$A_k = \{a_1, \dots, a_k\}$$

$$O_k = \{o_1, \dots, o_k\}$$

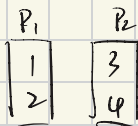
$$A_k + (O_k \setminus O_k) \in \mathcal{B}(M)$$

$$f(\text{Alg}) = \sum_{i=1}^k f(a_i | A_{i-1}) \geq \sum_{i=1}^k f(o_i | A_{i-1}) \geq \sum_{i=1}^k f(o_i | \text{Alg}) \geq f(\text{OPT} | \text{Alg})$$

(submod) (submod)

$$\Rightarrow f(\text{Alg}) \geq \frac{1}{e} \cdot f(\text{OPT})$$

tight example: partition matroid constraints



universe $U = \{x, y, z\}$

Coverage function: $S_1 = \{x\}$ $S_3 = S_4 = \{x, z\}$

$S_2 = \{y\}$

(weighted coverage): $w: U \rightarrow \mathbb{R}_{\geq 0}$

$$f(F) = w\left(\bigcup_{e \in F} e\right) \quad \forall F \subseteq E$$

Nonmonotone submodular maximization, (nonnegative, not normalized)

1. unconstrained

eg: cut function: $f(U) = |S(U)| \quad \forall U \subseteq V$ 

directed cut function: $f(U) = |S^+(U)| \quad \forall U \subseteq V$

cut: $\frac{1}{2}$ -approx by random sample. directed cut: $\frac{1}{4}$ -approx

Thm: Random sample gets $\frac{1}{4}$ -approx for nonmonotone submod

$\frac{1}{2}$ -approx for symmetric nonmonotone submod

Alg: Independently sample $y_i = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 0 & \text{w.p. } \frac{1}{2} \end{cases}$, $S = \{i: x_i = 1\}$

"Multilinear extension" $x \in [0, 1]^n$, let $R(x) \subseteq [n]$ be a random set where every i is sampled independently with prob. x_i $f(x) = \mathbb{E}(f(R(x)))$

$$f(x) = \sum_{S \subseteq [n]} \prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \cdot f(S)$$

Lemma: f is concave in direction $d \geq 0$ 

$$\frac{\partial f}{\partial x_i} = f(x \vee e_i) - f(x \wedge (1 - e_i)) = \mathbb{E}(f(R(x) \vee i)) - \mathbb{E}(f(R(x) \setminus i))$$

$\rightarrow$ multilinear

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial f(x \vee e_i)}{\partial x_j} - \frac{\partial f(x \wedge (1 - e_i))}{\partial x_j}$$

$$= f(x \vee e_i \vee e_j) - f(x \vee e_i \wedge \bar{e}_j) - (f(x \wedge \bar{e}_i \vee e_j) - f(x \wedge \bar{e}_i \wedge \bar{e}_j))$$

$$= \mathbb{E}(f(R \vee i \vee j) - f(R \vee i \setminus j) - f(R \setminus i \vee j) + f(R \setminus i \setminus j))$$

$$(\text{if } j \neq i) \leq 0$$

$$\nabla^2 f = \begin{pmatrix} 0 & & \\ & 0 & \\ - & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$d \in \mathbb{R}_+^n, t \in \mathbb{R}$$

$$g(t) = f(x + td)$$

$$g'(t) = d^T \nabla^2 f d \leq 0$$



pf of Thm:

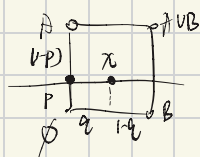
claim: For $A \subseteq [n]$, $0 \leq p \leq 1$, denote by $A(p)$ the random set where every $i \in A$ is sampled w.p. p independently. then

$$\mathbb{E}[f(A(p))] = \mathbb{E}[f(\mathbb{1}_A \cdot p)] \geq p \cdot f(A) + (1-p) \cdot f(\emptyset)$$

claim: For $p, q \in [0, 1]$, $\mathbb{E}[f(A(p) \cup B(q))] \geq (1-p)(1-q) \cdot f(\emptyset) + p(1-q) \cdot f(A) + (1-p)q \cdot f(B) + pq \cdot f(A \cup B)$

(let $g(B) := g(B \cup A(p))$)

$$\begin{aligned} \mathbb{E}[f(A(p) \cup B(q))] &\geq \mathbb{E}[q \cdot f(B \cup A(p)) + (1-q) \cdot f(A(p))] \\ &\geq q(p \cdot f(A \cup B) + (1-p) \cdot f(A)) + (1-q)(p \cdot f(A) + (1-p) \cdot f(\emptyset)) \\ &= \text{above} \end{aligned}$$



let $A = [n/2]$, $B = [n] \setminus [n/2] \Rightarrow A \upharpoonright q = A(\frac{1}{2}) \cup B(\frac{1}{2})$

$$\begin{aligned} \Rightarrow \mathbb{E}[f(A \upharpoonright q)] &\geq \frac{1}{4} f(\emptyset) + \frac{1}{4} f([n/2]) + \frac{1}{4} f([n/2]) + \frac{1}{4} f([n]) \\ &\geq \frac{1}{4} f([n/2]) + \frac{1}{4} f([n/2]) \} \geq \frac{1}{2} f([n/2]) \\ &= \frac{1}{2} f([n/2]) \text{ symmetric } f \end{aligned}$$

Double greedy:

deterministic alg:

$$A \leftarrow \emptyset, \bar{B} \leftarrow \emptyset$$

For $i = 1 \rightarrow n$
 $\alpha \leftarrow f(e_i | A)$
 $\beta \leftarrow \bar{f}(e_i | \bar{B})$

$$\text{if } \alpha \geq \beta : A \leftarrow A + e_i$$

$$\text{else } \bar{B} \leftarrow \bar{B} + e_i$$

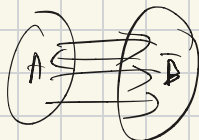
Output A (or equivalently $\bar{B}$)

Thm: $\frac{1}{2}$ -approx.

$$E = \{e_1, \dots, e_n\}$$

$$\text{let } \bar{f}(S) := f(E \setminus S)$$

Goal: A maximizes f
 $\bar{B}$ maximizes $\bar{f}$ ($\Leftrightarrow B$ maximizes f)



$$f(B + e) - f(B) = \bar{f}(\bar{B} \cup e) - \bar{f}(\bar{B}) = \bar{f}(e | \bar{B})$$

randomized alg.

$$A \in \mathcal{P}, \bar{B} \in \mathcal{P}$$

For $i=1, \dots, n$

$$\alpha_i \leftarrow \max\{f(e_i|A), 0\}$$

$$\beta_i \leftarrow \max\{\bar{f}(e_i|\bar{B}), 0\}$$

If $\alpha + \beta \neq 0$

$$\begin{cases} A \leftarrow A + e_i \text{ w.p. } \frac{\alpha_i}{\alpha_i + \beta_i} \\ \bar{B} \leftarrow \bar{B} + e_i \text{ w.p. } \frac{\beta_i}{\alpha_i + \beta_i} \end{cases}$$

else: $A \leftarrow A + e_i$

Output A (or equivalently B)

Analysis:

Lemma: Define $D_i := \text{OPT} \cup A_i \setminus \bar{B}_i$, then,

$$\mathbb{E}(f(D_{i+1}) - f(D_i)) \leq \frac{1}{2} \mathbb{E}(f(A_i) - f(A_{i-1}) + \bar{f}(\bar{B}_i) - \bar{f}(\bar{B}_{i-1})) \quad (*)$$

Then: Randomized double greedy gets $\frac{1}{2}$ -approx

pf: Since $D_0 = \text{OPT}$, $D_n = A$

$$\stackrel{\text{lemma}}{\Rightarrow} \mathbb{E}(f(\text{OPT}) - f(A)) \leq \frac{1}{2} \mathbb{E}(f(A) + \bar{f}(\bar{B})) = \mathbb{E}(f(A))$$

$$\Rightarrow \mathbb{E}(f(A)) \geq f(\text{OPT})$$

pf of lemma:

• $\alpha_i + \beta_i \neq 0$

$$\text{claim: } \mathbb{E}[f(A_i) - f(A_{i-1}) + \bar{f}(\bar{B}_i) - \bar{f}(\bar{B}_{i-1})] = \frac{\alpha_i^2 + \beta_i^2}{\alpha_i + \beta_i}$$

Condition on $A_{i-1}, \bar{B}_{i-1}$

pf: $\alpha_i, \beta_i \geq 0 \quad \checkmark$

Suppose $\alpha_i = 0, \beta_i > 0 \quad \checkmark \quad \square$

Case 1: $e_i \notin \text{OPT}$

f linear $\Rightarrow$ optimal

f symmetric

$$\mathbb{E}(f(x_{i-1}) - f(x_i)) = \mathbb{P}(e_i \in A_i) \cdot (-f(e_i | x_{i-1}))$$

$$\begin{aligned} A_i & \stackrel{\text{OPT}}{=} \left[\begin{array}{|c|} \hline \text{shaded region} \\ \hline \end{array} \right] \leq \frac{\alpha_i}{\alpha_i + \beta_i} \cdot (-f(e_i | B_{i-1} - e_i)) \\ \bar{B}_i & \stackrel{\text{OPT}}{=} \left[\begin{array}{|c|} \hline \text{shaded region} \\ \hline \end{array} \right] \leq \frac{\alpha_i}{\alpha_i + \beta_i} \cdot f(e_i | \bar{B}_{i-1}) \end{aligned} \quad (B_i = \bar{B}_i)$$

$$O_i = \text{OPT} \cup A_i \setminus \bar{B}_i \leq \frac{\alpha_i \beta_i}{\alpha_i + \beta_i}$$

Case 2: $e_i \in \text{OPT}$

$$\begin{aligned} \mathbb{E}(f(x_{i-1}) - f(x_i)) &= \mathbb{P}(e_i \in \bar{B}_i) \cdot f(e_i | x_{i-1} - e_i) \\ &\leq \frac{\beta_i}{\alpha_i + \beta_i} \cdot f(e_i | A_{i-1}) \\ &\leq \frac{\alpha_i \beta_i}{\alpha_i + \beta_i} \end{aligned}$$

$$\alpha_i^2 + \beta_i^2 \geq 2\alpha_i \beta_i \Rightarrow \text{lemma}$$

• $\alpha_i + \beta_i = 0$

Since $A_{i-1} \cap \bar{B}_{i-1} = \emptyset$, $A_{i-1} \subseteq B_{i-1} - e_i$

$$\Rightarrow f(e_i | A_{i-1}) \geq f(e_i | B_{i-1} - e_i) \Rightarrow f(e_i | A_{i-1}) + \bar{f}(\bar{e}_i | \bar{B}_{i-1}) \geq 0$$

$$\alpha_i + \beta_i = 0 \Rightarrow f(e_i | A_{i-1}) = \bar{f}(\bar{e}_i | \bar{B}_{i-1}) = 0 \Rightarrow \text{RHS of } (*) = 0$$

$$\text{LHS} = \begin{cases} -f(e_i | x_{i-1}) \leq \beta_i = 0 & \text{if } e_i \notin \text{OPT} \\ 0 & \text{if } e_i \in \text{OPT} \end{cases}$$



Continuous greedy:

Recall multilinear extension. $\bar{F}(x) = \mathbb{E}(F(R(x)))$

where $R(x)$ is a random set where $i \in [n]$ is sampled w.p. x_i

- ① $\bar{F}$ is monotone if f is monotone
- ② $\bar{F}$ is concave in direction $d \in \mathbb{R}_{\geq 0}^n$
- ③ $\bar{F}$ is convex in direction $e_i - e_j$.

Goal: $\max_{x \in P} f(x)$, where $P \subseteq [0,1]^n$ is some polytope

Alg: Fix some step size $S \ll 1$.

Initialize $x = 0$, $t = 0$

While $t < 1$.

$$v^* \leftarrow \operatorname{argmax}_{v \in P} \langle v, \nabla f(x) \rangle$$

$$x \leftarrow x + S \cdot v^*, \quad t \leftarrow t + S$$

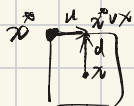
Return x

Remark: v^* is an extreme point of P .

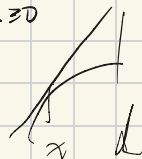
Final solution $x = \sum_i S_i v_i^*$ is a convex combination of extreme points of $P \Rightarrow x \in P$.

Thm: Continuous greedy gets $1 - \frac{1}{e}$

Analysis: Let $x^* = 1_{\text{opt}}$, $d := (x^* \vee x) - x = (x^* - x) \vee 0 \Rightarrow d \in \mathbb{R}^n_{\geq 0}$



$$x^* \geq d$$



$$\begin{aligned} \frac{d f(x(t))}{dt} &= \langle \nabla f(x), \frac{dx}{dt} \rangle = \langle \nabla f(x), v^* \rangle \stackrel{\text{(monotonicity)}}{\geq} \langle \nabla f(x), x^* \rangle \geq \langle \nabla f(x), d \rangle \\ &\stackrel{\text{(convexity)}}{\geq} f(x + d) - f(x) \stackrel{\text{(monotonicity)}}{\geq} f(x^*) - f(x) = f_{\text{opt}} - f(x(t)) \end{aligned}$$

$$\begin{aligned} \int_0^t \frac{d(e^{t'} f(x(t')))}{dt'} dt' &\geq \int_0^t e^{t'} f_{\text{opt}} dt' \Rightarrow e^t f(x(t)) \geq (e^t - 1) \cdot f_{\text{opt}} \\ &\Rightarrow f(x(t)) \geq (1 - e^{-t}) \cdot f_{\text{opt}} \end{aligned}$$

Nonmonoton submodular maximization (constrained)

$\max_{x \in P} f(x)$, where P is some downward-closed polytope.

Continuous greedy gets $(1 - \ln 2) \approx 0.307$.

Measured continuous greedy

Alg: Fix some step size $S < 1$.

Initialize $x = 0, t = 0$

While $t < 1$:

$$v^* \leftarrow \operatorname{argmax}_x \langle v, \nabla f(x) \rangle : v \in P, v \leq 1 - x$$

$$x \leftarrow x + S \cdot v^*, t \leftarrow t + S$$

Return x

Thm: Measured greedy gets $(1 - \frac{1}{e})$ for monotone, $\frac{1}{e}^{0.368}$ for nonmonotone

pf: monotone: $d = (x^* - x) \vee 0 \leq 1 - x \quad d \leq x^* \Rightarrow d \in P$

$$\Rightarrow \langle \nabla f(x), v^* \rangle \geq \langle \nabla f(x), d \rangle$$

nonmonotone:

lemma: if $x_i \in P \forall i \in [n]$, then $f(x \vee x^*) \geq (1-p) f(x^*)$

pf: let $u := (x \vee x^*) - x^* = (x - x^*) \vee 0 \in \mathbb{R}_{\geq 0}^d$

$$g(\lambda) := x^* + \lambda \cdot u$$

$$g(\frac{1}{p}) = x^* + \frac{1}{p} \cdot ((x - x^*) \vee 0) = (\frac{1}{p}x + (1 - \frac{1}{p})x^*) \vee x^* \leq (\frac{1}{p}x) \vee x^* \leq 1$$

$$f(x \vee x^*) = g(1) = g((1-p) \cdot 0 + p \cdot \frac{1}{p}) \geq \underline{(1-p) \cdot g(0)} + p \underline{g(\frac{1}{p})} \geq \underline{(1-p) \cdot f(x^*)}$$

lemma: $x_i(t) \leq 1 - e^{-t}$, $\forall i \in [n]$.

pf: $x'_i(t) = v_i \leq 1 - x_i(t) \Rightarrow x_i(t) \leq 1 - e^{-t}$

pf of Thm (nonmonotone): $\frac{d f(x(t))}{dt} \geq f(x \vee x^*) - f(x) \geq e^{-t} \cdot f(x^*) - f(x(t))$

$$\Rightarrow \int_0^t \frac{d(e^{-t} \cdot f(x(t)))}{dt} dt \geq \int_0^t f(x^*) dt \Rightarrow e^{-t} f(x(t)) \geq f(x^*)$$

i.e. $f(x(t)) \geq t \cdot e^{-t} \cdot f(\operatorname{opt})$

eg, $P = \{x: x_1 + \dots + x_n \leq k\}$

P matroid polytope.

Local Search.

Matroid-constrained Monotone submod max

$$\left[\begin{array}{l} k \text{ matroid intersection } \frac{1}{k} \rightarrow \frac{2}{k} [LSV] \\ \text{monotone submod s.t. } k \text{ matroids } \frac{1}{k+1} \rightarrow \frac{1}{k} [LSV] \\ \downarrow \\ \frac{2}{k+0.5k} [ST] \end{array} \right]$$

Alg: Initialize $S \leftarrow S_0$ basis

If $\exists a \in S, b \notin S, S-a+b \in \mathcal{B}(M)$, s.t. $f(S-a+b) > f(S)$
 $S \leftarrow S-a+b$.

Alg gets a local optimum S , i.e. $f(S') \leq f(S) \forall S': |S'| \leq |S|, |S \setminus S'| \leq 1$

Poly time Alg: swap if $f(S') > \frac{1}{1-\delta} f(S)$; $S_0 = \{v_i^*\}$, $i^* = \arg \max_i f(\{v_i\})$

Alg gets an $(1-\delta)$ -approx local optimum S , i.e. $f(S') \leq \frac{1}{1-\delta} f(S)$.

running time: S^* local optimum

$$\forall S': |S'| \leq |S|, |S \setminus S'| \leq 1$$

$$f(S_0) = f(v^*) \geq \frac{1}{n} f(S^*)$$

$$\log \left(\frac{f(S^*)}{\frac{1}{1-\delta} f(S_0)} \right) \leq \log_{\frac{1}{1-\delta}} n = - \frac{\log n}{\log(1-\delta)} \leq \frac{\log n}{\delta}$$

Lemma (Brualdi): $Alg - a_i + O_i \in \mathcal{B}(M)$

$$\begin{aligned} \text{Alg} \quad \text{OPT} \\ a_i \quad O_i \\ \downarrow \quad \downarrow \end{aligned} \quad f(Alg) = \sum_{i=1}^n f(a_i | A_{i-1}) \stackrel{(\text{submod})}{\geq} \sum_{i=1}^n f(a_i | Alg - a_i) \geq \sum_{i=1}^n f(O_i | Alg - a_i) \\ \geq \sum_{i=1}^n f(O_i | Alg) \stackrel{(\text{submod})}{\geq} f(OPT | Alg) \stackrel{(\text{monotone})}{\geq} f(OPT) - f(Alg)$$

Matroid constrained ^{nonnegative} submodular maximization (not necessarily monotone)

Alg: Initialize $S \leftarrow S_0 \in \mathcal{I}(M)$

If $\exists S' \in \mathcal{I}$, s.t. $|S'| \leq |S|, |S \setminus S'| \leq 1, f(S') > f(S)$
 $S \leftarrow S'$

Thm: Let $\alpha \geq 1$, $S, T, A_1, \dots, A_k \in E$, $\lambda_i \geq 0 \forall i \in [k]$

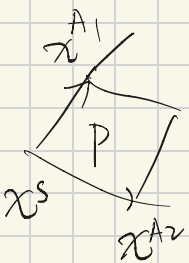
$$\text{If } x^T + (\alpha - 1) \cdot x^{S \cap T} = \alpha \cdot x^S + \sum_{i=1}^k \lambda_i (x^{A_i} - x^S)$$

$$\text{then } (\alpha + 1) \cdot f(S) + \sum_{i=1}^k \lambda_i (f(A_i) - f(S)) \geq f(S \cup T) + \alpha \cdot f(S \cap T)$$

Application: ① matroid constraint, monotone f .

$$\mathcal{F} := \{A \in \mathcal{B}(n) : \underbrace{d(A, S)}_{\substack{\parallel \\ |A \setminus S| = |S \setminus A|}} = 1\}$$

Fact: In matroid base polytope, $\mathbb{1}_A$ adjacent to $\mathbb{1}_S$ iff $A \in \mathcal{F}$



$$\Rightarrow \forall T: \exists \lambda_i \geq 0, \text{ s.t. } x^T = x^S + \sum_{A_i \in \mathcal{F}} \lambda_i \cdot (x^{A_i} - x^S)$$

$$S \text{ local optimum} \Rightarrow f(A_i) \leq f(S) \quad \forall A_i \in \mathcal{F} \quad (\text{monotone})$$

$$(\alpha \geq 1) \Rightarrow 2f(S) \geq f(S \cup T) + f(S \cap T) \geq f(T)$$

$$\text{pf of Thm: } \lambda := \sum \lambda_i \Rightarrow \sum \lambda_i x^{A_i} = x^T + (\alpha - 1) x^{S \cap T} + (1 - \alpha) x^S$$

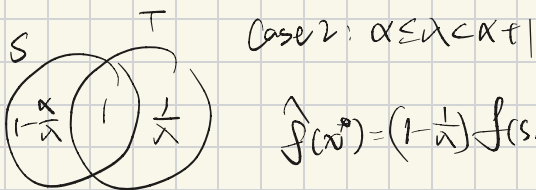
Let $\hat{f}$ be the Lovasz extension.

$$x^* := \frac{1}{\lambda} x^T + \left(\frac{\alpha}{\lambda} - \frac{1}{\lambda}\right) x^{S \cap T} + \left(1 - \frac{\alpha}{\lambda}\right) x^S$$

$$\sum \frac{\lambda_i}{\lambda} f(A_i) \geq \hat{f}\left(\sum \frac{\lambda_i}{\lambda} x^{A_i}\right) = \hat{f}(x^*)$$

$$\text{claim: } \hat{f}(x^*) \geq \frac{1}{\lambda} f(S \cup T) + \left(1 - \frac{\alpha}{\lambda} - \frac{1}{\lambda}\right) f(S) + \frac{\alpha}{\lambda} f(S \cap T) \Rightarrow \text{Thm } \checkmark$$

pf of claim: Case 1: $\lambda \geq \alpha + 1$ equality holds



$$\hat{f}(x^*) = (1 - \frac{1}{\lambda}) f(S \cap T) + (\frac{1}{\lambda} - 1 + \frac{\alpha}{\lambda}) f(T) + (1 - \frac{\alpha}{\lambda}) f(S \cup T)$$

$$\Leftrightarrow (\frac{1}{\lambda} - 1 + \frac{\alpha}{\lambda}) (f(S) + f(T)) \geq (\frac{\alpha}{\lambda} - 1 + \frac{1}{\lambda}) f(S \cap T) + (\frac{1}{\lambda} - 1 + \frac{\alpha}{\lambda}) f(S \cup T) \quad \checkmark$$

Case 3: $\lambda < \alpha \Rightarrow S \setminus T = \emptyset$ (Recall $x^* = \sum \frac{\lambda_i}{\lambda} x_i^*$)

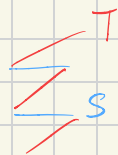
$$\hat{f}(x^*) = (1 - \frac{1}{\lambda}) f(S) + \frac{1}{\lambda} f(T)$$

$$\Leftrightarrow (1 - \frac{1}{\lambda}) f(S) + \frac{1}{\lambda} f(T) \geq \frac{1}{\lambda} f(T) + (1 - \frac{1}{\lambda}) f(S) \quad \checkmark$$

(k+1) matroid intersection $\Rightarrow$ submod maximization s.t. k matroids

Application: ② matching constraint, monotone f , P

S, S' matchings, $|S'| \geq (1+\epsilon) |S|$



Claim: $\exists$ S-augmenting path $P \subseteq S \Delta S'$, with length $\leq O(\frac{1}{\epsilon})$

pf: # S-augmenting path $\geq |S'| - |S|$

$$\text{Avg length} \leq \frac{|S| + |S'|}{|S'| - |S|} = 1 + \frac{2|S|}{|S'| - |S|} \leq 1 + \frac{2}{\epsilon}$$

$\Rightarrow$ local search over length $O(\frac{1}{\epsilon})$ augmenting path gives $(1+\epsilon)$ -

approx for maximum matching

How about submodular objective?

Lemma: $\forall$ matchings S, T : $\exists \Delta \geq 0$,

$$\text{s.t. } \chi^T + \varepsilon \cdot \chi^{S \cap T} = (1+\varepsilon) \chi^S + \sum_{A_i \in \mathcal{F}} \lambda_i \cdot (\chi^{A_i} - \chi^S),$$

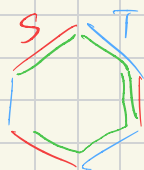
where $\mathcal{F} := \{A_i : |A_i \Delta S| \leq \frac{1}{\varepsilon}\}$.

pf: $h_i = \frac{1}{\varepsilon}$ Goal: Construct augmenting P_i , $|P_i \cap T| \leq h$, $|P_i \cap S| \leq h+1$

$$A_i := S \Delta P_i, \quad \sum \lambda_i (\chi^{A_i} - \chi^S) = \chi^{T \setminus S} - (1 + \frac{1}{h}) \cdot \chi^{S \cap T}.$$

Take $T \Delta S =$ collection of cycles & paths

cycles



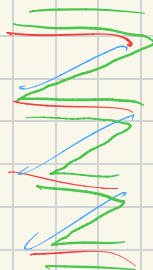
$h=1$

$P_i : |P_i \cap T| = h$

$\lambda_i = \frac{1}{h}$

h T-edges, $h+1$ S-edges

paths



✓

$$\Rightarrow \sum \lambda_i (\chi^{A_i} - \chi^S) + (1+\varepsilon) \chi^S = \chi^{T \setminus S} - (1+\varepsilon) \chi^{S \cap T} + (1+\varepsilon) \chi^S \\ = \chi^{T \setminus S} + (1+\varepsilon) \chi^{S \cap T} = \chi^T + \varepsilon \cdot \chi^{S \cap T}.$$

$$\text{Take } \alpha = 1+\varepsilon \text{ in Thm } \Rightarrow f(S) \geq \frac{1}{2+\varepsilon} f(S \cup T) + \frac{1+\varepsilon}{2+\varepsilon} f(S \cap T) \\ \text{(monotone)} \\ \geq \frac{1}{2+\varepsilon} f(LT).$$

Setting: $\mathcal{M} = (N, \mathcal{I})$

$$\{x \in [0, 1]^N : x(S) \leq r_{\mathcal{M}}(S) \forall S \subseteq N\}$$

(1)

$$\max \{F(x) : x \in P(\mathcal{M})\}$$

$$\text{multilinear relaxation} = \mathbb{E} \left(f(R(x)) = \sum_{i \in S} x_i \right) \left(\prod_{i \notin S} (1 - x_i) \right) f(S)$$

Recall: (1) Pick $d \in \mathbb{R}_{\geq 0}^N$, $g(t) = F(x + td)$

(2) $g(t)$ is concave in t

(3) For $d = e_i - e_j$, $h(t) = F(x + td)$ is convex for $\forall x \in [0, 1]^N$ in t

Thm: Efficient randomized rounding that outputs a random independent set $R \in \mathcal{I}$ (OBLIVIOUS to f !) (CVR, POC 2010)

$$a) \mathbb{E}(\mathbb{1}_R) = x$$

$$b) \mathbb{E}(f(R)) \geq F(x)$$

no loss rounding

Background: [Agarwal, Sivadasan, IPCO'99] deterministic rounding for bipartite matching polytope [CCPR, POC'07] $\rightarrow$ matroid polytope F multilinear ext when $f = \text{sum of wtd matroid rank fns}$ (1-1/e) approx $\rightarrow$ [Vondrak'08] CG

Recap: Basis exchange property $\Rightarrow \mathcal{X}_{B_1}$ & $\mathcal{X}_{B_2}$ connected by an edge in 01 $P(\mathcal{M})$ iff $|B_1 \Delta B_2| = 2$

(2) $x \in P(\mathcal{M})$ and A, B tight sets w.r.t x ($x(A) = r_{\mathcal{M}}(A)$) then $A \cap B$ & $A \cup B$ also tight

$$[x(A) + x(B) = r(A) + r(B) \geq r(A \cup B) + r(A \cap B) \geq x(A \cup B) + x(A \cap B) = x(A) + x(B)]$$

Lemma: Let $\tau \in \mathbb{N}$, $x_t \in [0, 1]^N$ for $t = \{0, \dots, \tau\}$ satisfies

$$1) \forall t \in [\tau] \exists \text{ pair of coordinates } i, j : x_{t+1} = x_t + \xi(e_i - e_j)$$

$$2) \text{ conditioned on changing } i, j \quad \mathbb{E}[x_{t+1} | x_t, i, j] = x_t$$

then for any submodular fn $f : \{0, 1\}^N \rightarrow \mathbb{R}$ & its multilinear ext F

for any $t \in [\tau]$

$$\mathbb{E}(F(x_{t+1}) | x_t) \geq F(x_t)$$

(submartingale)

Pf. Since $x_{t+1} = x_t + \xi(e_i - e_j)$ (2)
 $\triangleright E[x_{t+1} | x_t, i, j] = x_t \Rightarrow E[\xi] = 0$
 $\phi(\xi) = F(x_t + \xi(e_i - e_j))$ is convex in ξ
 $E[F(x_{t+1}) | x_t, i, j] = E[\phi(\xi)] \geq \phi(0) = F(x_t)$ □

For $x \in B(N)$ need to find $e_i \neq e_j$ & find ξ to make str. poly progress

Randomized Pipage Rounding

If x_t not integral since $x_t(N) = x_t(N)$, ≥ 2 fractional variables i, j

$$\text{Set } x_{t+1} = \begin{cases} x_t + \delta(e_i - e_j) & \text{w/ pr } \delta / (\delta + \delta') \\ x_t + \delta'(e_j - e_i) & \text{w/ pr } \delta' / (\delta + \delta') \end{cases}$$

How big can δ be?

$$\delta = \min_{\substack{S \subseteq X \\ S \ni i, S \not\ni j}} (x_N(S) - x_t(S))$$

Similarly for δ'

Special case of submodular for minz

Progress? If one more variable integral $\checkmark$ restart with $x_{t+1}(N)$

else, we found $S \ni i, S \not\ni j$ nearly tight for

$$x_{t+1} = x_t + \delta(e_i - e_j) \text{ say}$$

But this means by (2), S, T tight $\Rightarrow S \cap T$ tight and still contains fractional x @ i , continue with SNT.

Detail: When looking for δ, δ' , we may have strictly smaller ^{tight} sets containing exactly i or exactly j (not both), in which case we use (2) to get even smaller sets.

(3)

Algo: Pipage Round(x)

While x not integral

$T \leftarrow N$.

While (T has fractional vars)

Pick i, j fractional $\in T$

$(x^+, A^+) \leftarrow \text{HitConstraint}(x, i, j)$

$(x^-, A^-) \leftarrow \text{HitConstraint}(x, j, i)$

If $(x^+ = x^- = x)$ then $T \leftarrow T \cap A^+ \text{ (or } T \cap A^-)$ $\delta = \delta' = 0$

else $p \leftarrow \|x^+ - x\| / \|x^+ - x^-\|$

With prob p , set $x \leftarrow x^-$, $T \leftarrow T \cap A^-$

prob $(1-p)$ set $x \leftarrow x^+$, $T \leftarrow T \cap A^+$

Output x

HitConstraint(x, i, j)

$A = \{A \subseteq X : i \in A, j \notin A\}$

Find $\delta = \min \{x_M(A) - x(A) : A \in A\}$ and A^* the argmin

If $x_j < \delta$ set $\{x_i \leftarrow x_i + \delta, x_j \leftarrow 0, A' \leftarrow \{j\}\}$

else $\{x_i \leftarrow x_i - \delta, x_j \leftarrow x_j + \delta, A' \leftarrow A\}$

Return (x, A')

Thm
Lemma Given $x_0 \in B(M)$ PipageRnd(x_0) outputs in poly time a random base $R \in B(M)$: $E(1_R) = x_0$ & $E(f(R)) \geq f(x_0)$.

PF: Once variables hit 0 or 1, they don't change.

Algo keeps tightening T until it's tight to x within $B(M)$.

Increasing x_i by δ made sure we satisfy $x(S) \leq n(M) \forall S \ni i$

Initially $T \leftarrow N$ tight. Moving to $T \cap A^+$ or $T \cap A^-$ keeps

the next T tight by 0(2).

If we don't make a new var integral, $|T|$ decreases, can happen $\leq n-1$ times

(4)

Since $E[x_{t+1} | x_t, i, j] = x_t$, lemma 1 gives $E(\|R\|) \geq F(x_0) \square$

Pipeline starts @ x & hits faces moving to lower dimensions

Randomized Swap Rounding (RSR) starts with ~~vector~~^{convex} decomposition into integer bases $x = \sum_{i \in I} \alpha_i \mathbf{1}_i$ $\alpha_i \geq 0$ $\sum_i \alpha_i = 1$

Picking R using this prob distribn satisfies $E(\|R\|) = x$ but not $E(F(R)) \geq F(x)$

Idea: Use symmetric exchange lemma

Say $x_0 = \sum_{i=1}^R \beta_i \mathbf{1}_{B_i}$

say $x_0 = \sum_{i=1}^R \beta_i \mathbf{1}_{B_i}$

Merge B_1 & B_2 into a new C_2

Pick $i \in B_1 \setminus B_2$ & $j \in B_2 \setminus B_1$ st. $B_2 + e_i - e_j$ & $B_1 - e_i + e_j$ are bases

Move to one of them w/ prob

$\beta_1 / \beta_1 + \beta_2$ $\beta_2 / \beta_1 + \beta_2$

Then merge C_2 with $B_3 \dots$ in $(n-1)$ steps to get a single base.

Algo RSR ($x = \sum_{i=1}^R \beta_i \mathbf{1}_{B_i}$)

$C_1 \leftarrow B_1$

For $k = 1$ to $(n-1)$

$C_{k+1} \leftarrow \text{Merge Bases } (\sum_{i=1}^R \beta_i, C_k, \beta_{k+1}, B_{k+1})$

Output C_n

Merge Bases $(\beta_1, B_1, \beta_2, B_2)$

While $(B_1 \neq B_2)$

Pick $i \in B_1 \setminus B_2$ & $j \in B_2 \setminus B_1$ st. $\{B_1 - e_i + e_j, B_2 - e_j + e_i\}$ are bases

w/ probability $\beta_1 / \beta_1 + \beta_2$ set $B_2 \leftarrow B_2 - e_j + e_i$
else $B_1 \leftarrow B_1 - e_i + e_j$

Output B_1

(5)

Merge Bases runs for $\leq |B_i| = d$ steps $\Rightarrow \leq nd$ steps
 RSR runs for $\leq |N| = n$ rounds

Need to decompose x_0 into convex comb of n bases by
 Carathéodory's thm $O(n^d)$ calls to independence oracle [Example 8.1]
 d^2 calls to ~~merge~~ find ij pairs to swap $\Rightarrow nd^2$ calls for algo.

Thm 3 Given $x_0 \in \mathcal{B}(N)$, $\text{RSR}(x_0)$ outputs in poly time a random
 base $R \in \mathcal{B}(N)$ st $E(1_R) = x_0$ & $E(f(R)) \geq F(x_0)$

Pf: Check lemma 1: 1) change in e_i & e_j same for each merge
 step

2) Swap moves B_1, B_2 to B_1', B_2'

$$\begin{aligned} B_1' &= B_1 \quad \left(\text{w/prob } \frac{\beta_1}{\beta_1 + \beta_2} \right) &= B_1 - i + j \quad \left(\text{w/prob } \frac{\beta_2}{\beta_1 + \beta_2} \right) \\ B_2' &= B_2 - j + i \quad \left(\text{w/prob } \frac{\beta_1}{\beta_1 + \beta_2} \right) &= B_2 \quad \left(\text{w/prob } \frac{\beta_2}{\beta_1 + \beta_2} \right) \end{aligned}$$

$$E(\beta_1 1_{B_1} + \beta_2 1_{B_2} | x_{t,i,j}) = \frac{\beta_1}{\beta_1 + \beta_2} (\beta_1 1_{B_1} + \beta_2 (1_{B_2} - e_i + e_j))$$

$$+ \frac{\beta_2}{\beta_2 + \beta_1} (\beta_1 1_{B_1 - e_i + e_j} + \beta_2 1_{B_2})$$

$$= \frac{\beta_1}{\beta_1 + \beta_2} 1_{B_1 + B_2} + \frac{\beta_2}{\beta_1 + \beta_2} 1_{B_1 + B_2}$$

$$\text{So } E(x_{t+1} | x_{t,i,j}) = E(\beta_1 1_{B_1'} + \beta_2 1_{B_2'} + \sum_{l=3}^R \beta_l 1_{B_l})$$

$$= x_t \quad \text{Lemma 1 implies Thm 3} \quad \square$$

Run pictorially on sp tree polytope

$$x(E[Y]) = n-1$$

$$x(E[S]) \leq |S| - 1 \quad \forall S \subseteq V$$

Very different algorithms

(8)

Properties
Addnl Implications:

Thm 3 If $R = (X_1, X_2, \dots, X_n)$ obtained by Pipeage or RSR,

The set $X_1, \dots, X_n$ are negatively correlated
of rws $E(\prod_{i \in T} X_i) \leq \prod_{i \in T} E(X_i) \quad \forall T \subseteq [n]$
 $E(\prod_{i \in T} (1 - X_i)) \leq \prod_{i \in T} (1 - E(X_i))$

This implies Chernoff-type bounds:

For any linear $Z = \sum_i a_i X_i$ with $a_i \in [0, 1]$
 $E(Z) = \sum_i a_i E(X_i)$

Upper tail: 1) If $\delta \geq 0$ & $\mu \geq E(Z)$, $P_n(Z \geq (1+\delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$
Lower tail: 2) If $\delta \in [0, 1]$ & $\mu \leq E(Z)$, $P_n(Z \leq (1-\delta)\mu) \leq e^{-\frac{\mu\delta^2}{2}}$

Pr: $f_T(s) = 1 \iff S \cap T \neq \emptyset$
= 0 o/w

$$F_T(\mathbf{x}) = 1 - \prod_{i \in T} (1 - x_i)$$

$$1 - E(\prod_{i \in T} (1 - X_i)) = E(f_T(X_1, \dots, X_n)) \rightarrow \text{defn of } f_T$$

$$\geq F_T(x_1, \dots, x_n)$$

$$= 1 - \prod_{i \in T} (1 - x_i)$$

$$= 1 - \prod_{i \in T} (1 - E(X_i)) \rightarrow \text{by bounding guarantees}$$

To get $E(\prod_{i \in T} X_i) \leq \prod_{i \in T} E(X_i)$

use function $f'_T = f_T(NIS)$.

~~if $\delta \geq 0$ & $\mu \geq E(Z)$~~

Negative correln

$\Rightarrow$ tail bounds [Pancoski-Srinivasan '97]

□

Nonoblivious local search: [FW13]

Monotone sub max under matroid constraint.

right example for local search: Coverage function, s.t. partition method.

$$\frac{\text{Alg}}{\text{opt}} = \frac{1}{2} \quad \text{Alg} \left[\frac{f(x)}{f(y)} \right] \quad \left[\frac{f(y)}{f(x)} \right]$$

f : multilinear extension.

Auxiliary function: $\tilde{f}(s) = \int_0^1 f(p \cdot \mathbb{1}_s) \cdot \frac{e^p}{p} dp$

Note: $f(p \cdot \mathbb{1}_s) = p \sum_{i \in s} f(i) + 0(p) = O(p) \quad (p \rightarrow 0) \Rightarrow$ well defined

Alg. find a local optimum of $\tilde{f}$.

Thm. $(1 - \frac{1}{e})$ -approx

pf: let $A := \text{Alg}$, $O := \text{opt}$

Lemma 1 $\exists h: A \setminus O \rightarrow O \setminus A$ bijection, s.t. $A - a + h(a) \in B \quad \forall a \in A$

Lemma 2 $\sum_{a \in A \setminus O} \tilde{f}(a | A - a) \geq \sum_{a \in O \setminus A} \tilde{f}(a | A)$

$$\begin{aligned} \text{pf: } \sum_{a \in A \setminus O} \tilde{f}(a | A - a) &= \sum_{a \in A \setminus O} (\tilde{f}(A) - \tilde{f}(A - a)) \geq \sum_{a \in A \setminus O} (\tilde{f}(A - a + h(a)) - \tilde{f}(A - a)) \\ &= \sum_{a \in A \setminus O} \tilde{f}(h(a) | A - a) \geq \sum_{a \in A \setminus O} \tilde{f}(h(a) | A) = \sum_{a \in O \setminus A} \tilde{f}(a | A) \end{aligned}$$

Lemma 3 $\sum_{a \in B} \nabla_a f(x) \geq f(x \vee \mathbb{1}_B) - f(x)$

pf: let $x^* = \mathbb{1}_B$, $d := (x \vee x^*) - x = 0 \vee (x^* - x) \Rightarrow 0 \leq d \leq x^*$

$$f(x \vee x^*) - f(x) = f(x + d) - f(x) \leq \langle \nabla f(x), d \rangle \leq \langle \nabla f(x), x^* \rangle = \sum_{j \in B} \nabla_j f(x)$$

(Concavity)

pf of thm: Lemma 2 $\Rightarrow \sum_{a \in O \setminus A} \tilde{f}(a | A) + \sum_{a \in O \setminus A} \tilde{f}(a | A - a) \geq \sum_{a \in A} \tilde{f}(a | A - a)$

$$\sum_{a \in O \setminus A} \tilde{f}(a | A) = \int_0^1 \sum_{a \in O \setminus A} \frac{f(p \cdot \mathbb{1}_{A+a}) - f(p \cdot \mathbb{1}_A)}{p} \cdot e^p dp$$

$$= \int_0^1 \sum_{a \in O \setminus A} \nabla_a f(p \cdot \mathbb{1}_A) \cdot e^p dp$$

$$\sum_{a \in \Omega A} \tilde{f}(a|A-a) = \int_0^1 \sum_{a \in \Omega A} \frac{F(p \cdot \mathbb{1}_A) - F(p \cdot \mathbb{1}_{A-a})}{p} \cdot e^p dp$$

$$= \int_0^1 \sum_{a \in \Omega A} \nabla_a F(p \cdot \mathbb{1}_A) \cdot e^p dp$$

$$\Rightarrow \sum_{a \in \Omega A} \tilde{f}(a|A) + \sum_{a \in \Omega A} \tilde{f}(a|A-a) = \int_0^1 \sum_{a \in \Omega} \nabla_a F(p \cdot \mathbb{1}_A) \cdot e^p dp$$

$$\geq \int_0^1 (F(p \cdot \mathbb{1}_A \vee \mathbb{1}_0) - F(p \cdot \mathbb{1}_A)) \cdot e^p dp \geq \int_0^1 (f_{\text{OPT}} - F(p \cdot \mathbb{1}_A)) \cdot e^p dp$$

$$\sum_{a \in A} \tilde{f}(a|A-a) = \int_0^1 \sum_{a \in A} \frac{F(p \cdot \mathbb{1}_A) - F(p \cdot \mathbb{1}_{A-a})}{p} \cdot e^p dp = \underbrace{\int_0^1 \sum_{a \in A} \nabla F(p \cdot \mathbb{1}_A) \cdot e^p dp}_{\int_0^1 g'(p) e^p dp}$$

$$g(p) := F(p \cdot \mathbb{1}_A)$$

$$\Rightarrow \int_0^1 g'(p) \cdot e^p dp \geq (e-1) \cdot f_{\text{OPT}} - \int_0^1 g(p) e^p dp$$

$$\Rightarrow g(1) \cdot e^1 - g(0) \cdot e^0 \geq (e-1) \cdot f_{\text{OPT}}$$

$$\Rightarrow f_{\text{ALG}} \geq (1 - \frac{1}{e}) f_{\text{OPT}}$$

Poisson process. [GKSS 26].

Recall continuous greedy: $x(t+s) = x(t) + s \cdot z(t) \Leftrightarrow \frac{x(t+s)}{t+s} = \frac{x(t)}{t+s} + \frac{s \cdot z(t)}{t+s}$

where $z(t) \in \arg \max \{ \langle \nabla F(x), v \rangle : v \in p(\mathcal{M}) \}$

$z(t)$ is indicator vector of a basis.

$$A(t) := \frac{x(t)}{t} \Leftrightarrow A(t+s) = \frac{t}{t+s} A(t) + \frac{s}{t+s} z(t).$$

Round on the fly: Given a basis $A(t)$, $A(t+s) = \begin{cases} A(t) & \text{w.p. } \frac{t}{t+s} \\ z(t) & \text{w.p. } \frac{s}{t+s} \end{cases}$

Alg: Initialize $A \in \mathcal{B}(m)$, $r \in \mathbb{R}^m$

Poisson clock with rate $\lambda_t = \frac{r}{t}$ from $t \in [0, 1]$

If event occurs:

$$\text{pick } Z \in \arg\max_{B \in \mathcal{B}(M)} \sum_{i \in B} \nabla F(t) \mathbb{1}_A$$

Find bijection $h: A \rightarrow \mathbb{Z}$, s.t. $A - a + h(a) \in \mathcal{B}(M)$, $h(a) = a \ \forall a \in A \cap S$

Draw a uniformly random a from A and $A \in A - a + h(a)$.

thm: Alg gets $(1 - \frac{1}{e})$ -approx.

$$\text{pf: Poisson distribution: } P(X=k) = \frac{e^{-\lambda S} (\lambda S)^k}{k!}$$

$$\Rightarrow P(X=0) = e^{-\lambda S} = 1 - \lambda S + \frac{(\lambda S)^2}{2} - \dots = 1 - \lambda S + o(S)$$

$$P(X=1) = e^{-\lambda S} (\lambda S) = \lambda S + o(S)$$

$$P(X \geq 2) = o(S)$$

$$g(t) := F(t, \mathbb{1}_{A(t)})$$

$$g'(t) = \lim_{S \rightarrow 0} \frac{g(t+S) - g(t)}{S} = \sum_{a \in A(t)} \nabla_a F(t, \mathbb{1}_{A(t)}) + \lim_{S \rightarrow 0} \frac{(1-\lambda S) \cdot F(t, \mathbb{1}_{A(t)}) + \frac{\lambda S}{r} \sum_{a \in A(t)} F(t, \mathbb{1}_{A(t)-a+h(a)}) - F(t, \mathbb{1}_{A(t)})}{S}$$

$$= \sum_{a \in A(t)} \nabla_a F(t, \mathbb{1}_{A(t)}) + \frac{\lambda}{r} \sum_{a \in A(t)} (F(t, \mathbb{1}_{A(t)-a+h(a)}) - F(t, \mathbb{1}_{A(t)}))$$

$$= \sum_{a \in A(t)} \nabla_a F(t, \mathbb{1}_{A(t)}) + \frac{\lambda}{r} \sum_{a \in A(t)} (F(t, \mathbb{1}_{A(t)-a+h(a)}) - F(t, \mathbb{1}_{A(t)-a}) - (F(t, \mathbb{1}_{A(t)}) - F(t, \mathbb{1}_{A(t)-a})))$$

$$= \sum_{a \in A(t)} \nabla_a F(t, \mathbb{1}_{A(t)}) + \frac{\lambda t}{r} \sum_{a \in A(t)} (\nabla_{h(a)} F(t, \mathbb{1}_{A(t)-a}) - \nabla_a F(t, \mathbb{1}_{A(t)}))$$

$$\geq \sum_{a \in A(t)} \nabla_a F(t, \mathbb{1}_{A(t)}) + \frac{\lambda t}{r} \sum_{a \in A(t)} (\nabla_{h(a)} F(t, \mathbb{1}_{A(t)}) - \nabla_a F(t, \mathbb{1}_{A(t)}))$$

$$= \sum_{a \in \mathbb{Z}} \nabla_a F(t, \mathbb{1}_{A(t)})$$

$$\geq \sum_{a \in \text{OPT}} \nabla_a F(t, \mathbb{1}_{A(t)})$$

$$\geq F(t, \mathbb{1}_{A(t)} \vee \mathbb{1}_{\text{OPT}}) - F(t, \mathbb{1}_{A(t)})$$

$$\geq f_{\text{OPT}} - g(t)$$

$$\Rightarrow (g(t) \cdot e^t)' = e^t f_{\text{OPT}} \Rightarrow g(t) \cdot e \geq (e-1) f_{\text{OPT}} \Rightarrow f_{\text{ALG}} \geq (1 - \frac{1}{e}) f_{\text{OPT}}$$