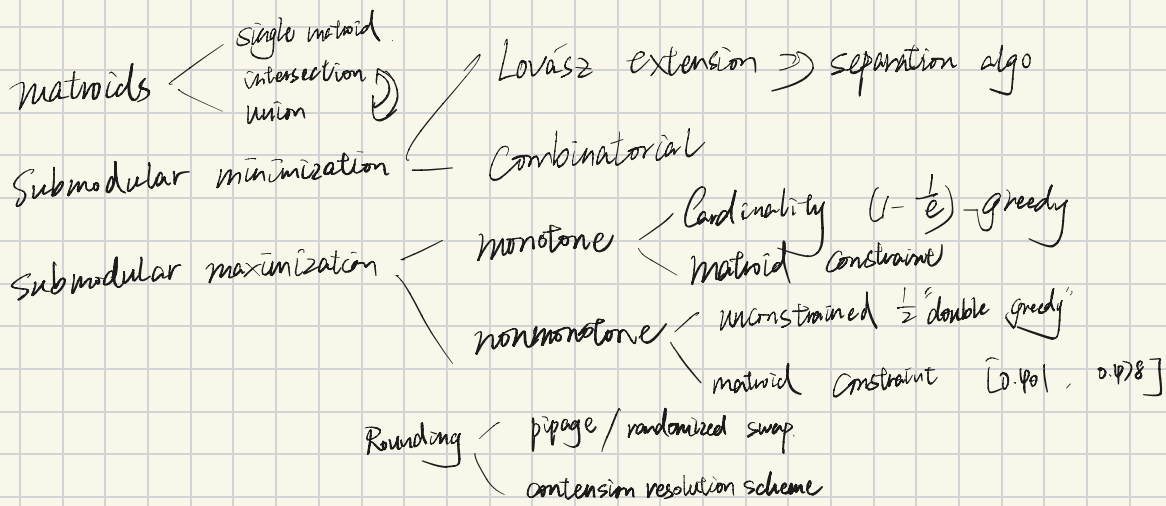




groundset $\uparrow$ independent sets.

Matroids $M = (E, \mathcal{I})$

1. Independent set axiom:

① (Downward-closedness): $I \in \mathcal{I}, J \subseteq I \Rightarrow J \in \mathcal{I}$

② (Extension): $I, J \in \mathcal{I}, |J| < |I| \Rightarrow \exists e \in I \setminus J, \text{ s.t. } J \cup e \in \mathcal{I}.$

Cor: Maximum ind sets have the same size.
 $\downarrow$
 Bases rank.

eg: linear matroids, graphic matroids

2. Basis axiom:

① (Exchange): $\forall B, D \in \mathcal{B}(M)$, $e \in B \setminus D$: $\exists f \in D \setminus B$, s.t. $B - e + f \in \mathcal{B}(M)$

(Exchange):

$f \in D \setminus B$: $\exists e \in B \setminus D$ — — — — —

$\uparrow$
 Only require f is not self-loop (singleton dependent set)

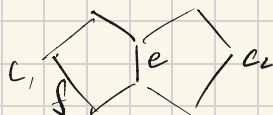
3. Circuit axiom:

def: A circuit is an inclusion-wise minimal dependent set.

① $C_1, C_2 \in \mathcal{C}(M), C_1 \subseteq C_2 \Rightarrow C_1 = C_2$

② $C_1 \neq C_2 \in \mathcal{C}(M), e \in C_1 \cap C_2 \Rightarrow \exists C \in \mathcal{C}(M), \text{ s.t. } C \subseteq (C_1 \cup C_2) \setminus \{e\}.$

pf: Need to show: $(C_1 \cup C_2) \setminus \{e\}$ dependent. Suppose not. i.e. $(C_1 \cup C_2) \setminus \{e\} \in \mathcal{I}$



pick arbitrary $f \in C_1 \setminus C_2 \Rightarrow C_1 \setminus f \in \mathcal{I}$

(Extension) $C_1 \cup C_2 \setminus \{f\} \in \mathcal{I}$ contradiction as $C_2 \subseteq (C_1 \cup C_2) \setminus \{f\}$

Cor: (unique circuit): Suppose $I \in \mathcal{I}, e \notin I$ and $I + e \notin \mathcal{I}$, then $\exists$ a unique circuit containing e .

= "fundamental circuit" if $I \in \mathcal{B}(M)$, denoted as $C(e, I)$.

$B - e + f \in \mathcal{B}(M) \Leftrightarrow e \in C(f, B)$.

4. Rank axiom

Def: $r(X) := \max \{|I| : I \in \mathcal{I}, I \subseteq X\}$

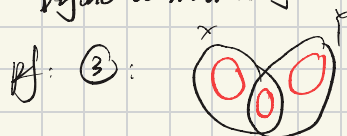
$$r: 2^E \rightarrow \mathbb{Z}_{\geq 0}$$

$$\textcircled{1} 0 \leq r(X) \leq |X|$$

$$\textcircled{2} (\text{monotonicity}) X \subseteq Y \Rightarrow r(X) \leq r(Y)$$

$$\textcircled{3} (\text{submodularity}): r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$$

Define a matroid from rank function: $\mathcal{I} := \{I \subseteq E : r(I) = |I|\}$



$$I_X \cap Y \subseteq I_X \subseteq I_X \cup Y \quad (\text{extension})$$

I_X is some max ind set of S .

$$r(Y) \geq |I_X \cup Y \setminus I_X| + |I_X \cap Y| = r(X \cup Y) - r(X) + r(X \cap Y)$$

eg: transversal matroid.

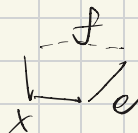
1. closure axiom "span": Def: $cl(X) := \{e \in E : r(X+e) = r(X)\}$

$$\textcircled{1} X \subseteq cl(X)$$

$$\textcircled{2} X \subseteq Y \Rightarrow cl(X) \subseteq cl(Y)$$

$$\textcircled{3} cl(cl(X)) = cl(X)$$

$$\textcircled{4} f \in cl(X+e) \setminus cl(X) \Rightarrow e \in cl(X+f)$$



pf: $\textcircled{2} e \in cl(X) \Rightarrow r(e|X) = 0 \Rightarrow r(e|Y) \leq r(e|X) = 0 \Rightarrow e \in cl(Y)$

$$\textcircled{3} Y := cl(X) \quad \forall e \in Y: r(e|Y) = r(Y) = r(X) \geq r(e|X)$$

$$\textcircled{4} r(f|X+e) = 0, r(f|X) = 1 \Rightarrow r(f|X+f) = r(f|X+e) - r(f|X) + r(e|X)$$

Lemma: For $e \notin X$, we have $e \in cl(X) \Leftrightarrow \exists C \in \mathcal{C}(X): e \in C \subseteq X+e$

closed set: "flat"

maximum proper flat: hyperplane

Deletion $\sum_{\text{minor}}$
Contraction

Dual matroid
generator

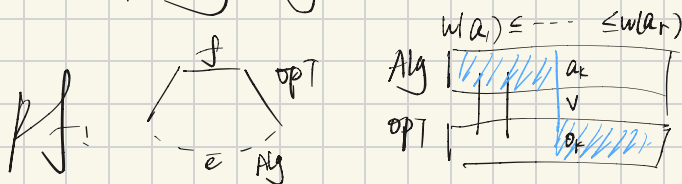
Direct sum

cocircuit:

Truncation

minimal "blocking" set of $\mathcal{B}(M)$
cuts & graphic matroids.

Greedy alg: sort $w_1 \leq \dots \leq w_n$,



Assume f_{SOC} $w(OPT) < w(Alg)$

$$B_i := OPT + \{a_{i-1}, a_i\} - \{o_1, \dots, o_i\} \in \mathcal{B}(M) \quad \forall i \in [r]$$

$$\exists w(a_k) > w(o_k) \quad Alg_{k-1} + o_k \leq B_{k-1} \rightarrow \leftarrow$$

Basis exchange

Serial exchange

Stronger exchange properties:

Thm (Symmetric exchange):

pf by closure

Conj (Gabow): Serial + symmetric exchange.

Thm (Borodin: matching):

pf by closure + Hall

Matroid intersection: [eg: bipartite matching, arborescence]

Thm: $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$ matroids.

$$\max \{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min \{r_1(s) + r_2(E \setminus s) \mid s \subseteq E\}$$

pf: " $\leq$ " $|I| = |I \cap s| + |I \cap (E \setminus s)| \leq r_1(s) + r_2(E \setminus s) \quad \forall s \subseteq E$.

" $\geq$ " Alg:  Exchange graph:
 $I + f - e \in \mathcal{I}_1 \Leftrightarrow (f, e) \in D$
 $I + f - e \in \mathcal{I}_2 \Leftrightarrow (e, f) \in D$
 $S := \{f : I + f \in \mathcal{I}_2\}$
 $T := \{f : I + f \in \mathcal{I}_1\}$

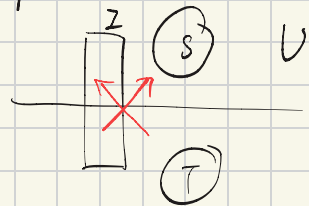
Find a shortest augmenting path P from s to t

Augment along P

Analysis: ① Existence of augmenting path:

Suppose not: $\exists$ disjoint $S, T \subseteq U$

s.t. $S \subseteq U$, $U \cap T = \emptyset$



$\mathcal{I}_1 \cap M_1$, $U \in \mathcal{C}(I \cap U)$

$\mathcal{I}_2 \cap M_2$, $E \setminus U \in \mathcal{C}(I \setminus U)$

$$\Rightarrow r_1(U) + r_2(E \setminus U) = |I \cap U| + |I \setminus U| = |I|$$

$\Rightarrow I$ is maximum common ind set.

$$\textcircled{2} I \Delta P \in \mathcal{I}_1 \cap \mathcal{I}_2$$

Lemma 1 (unique perfect matching): Matroid $M = (E, \mathcal{I})$, $I \in \mathcal{I}$,

$J \in \mathcal{E}$, s.e. $|I| = |J|$. If $D_n(I) \cap \mathcal{E}$ contains a unique PM, then $I \cap J \in \mathcal{I}$.



Lemma 2: Bipartite graph $G = (I \cup J, E)$, $|I| = |J|$.

$\exists$ unique PM $\Leftrightarrow \exists$ ordering $I = \{i_1, \dots, i_n\}$, $J = \{j_1, \dots, j_n\}$.

s.e. $N(\{i_1, \dots, i_k\}) = \{j_1, \dots, j_k\}$.

pf: " $\Leftarrow$ " $(i_1, j_1), \dots, (i_n, j_n)$ is the unique PM.

$\Rightarrow$ Induction on k



Say M is the unique PM.

$N(I \setminus k) = J \setminus k \Rightarrow I \setminus k$ is matched to $J \setminus k$

Let $G' = G \setminus (I' \cup J')$ where $I' = I \setminus I \setminus k$, $J' = J \setminus J \setminus k$.

Contract M in $G' \Rightarrow$ forest on $n' = n - k$ vertices

$P := E(G') \setminus M \Rightarrow \sum_{v \in I'} \deg_P(v) = |I'| - 1 \Rightarrow \exists v \in I'$ s.t. $\deg_P(v) = 0$

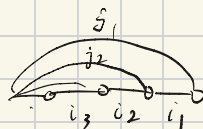
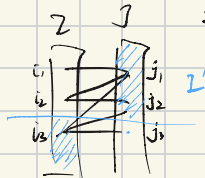
Let $i_k = v$ and $j_k \in J$ be the vertex matched to i_k in M .

$\Rightarrow N(I \setminus k) = J \setminus k \quad \square$

pf of unique PM Lemma.

Lemma 2 $\Rightarrow I \setminus J = \{i_1, \dots, i_n\}$, $J \setminus I = \{j_1, \dots, j_n\}$

s.e. $N(\{i_1, \dots, i_k\}) = \{j_1, \dots, j_k\} \quad \forall k \in n$



claim: $I \setminus I \setminus k + J \setminus k \in \mathcal{I}$

Ex: Grabow

pf. Induction. Suppose $\exists \text{SOC } I - I_{ck} + J_{ck} \in \mathcal{I}$
 $I - I_{ck} + J_{ck} \notin \mathcal{I}$

$\Rightarrow \exists$ circuit $C = C(j_k, I')$, $j_k \notin C$, where $I' := I - I_{ck} + J_{ck}$.

let $C' := C(j_k, I) \Rightarrow C' \notin \mathcal{C}$

circuit axiom $\Rightarrow \exists C' \subseteq C \cup C'$, $j_k \notin C'$

$\Rightarrow C' \subseteq I'$ contradiction.

③ Termination: $|I|$ increases every iteration.

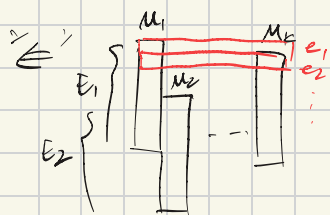
(Weighted matroid intersection)^{*}

Matroid union: $M_1 = (E_1, \mathcal{I}_1), \dots, M_k = (E_k, \mathcal{I}_k)$

$\tilde{M} = M_1 \vee M_2 \vee \dots \vee M_k = (E_1 \cup \dots \cup E_k, \mathcal{I}_1 \cup \dots \cup \mathcal{I}_k)$

where $I \in \mathcal{I}_1 \cup \dots \cup \mathcal{I}_k$ if $I \subseteq I_1 \cup \dots \cup I_k$, s.t. $I_j \in \mathcal{I}_j$.

Matroid union $\Leftrightarrow$ Matroid intersection.



$M' = M_1 \oplus M_2 \oplus \dots \oplus M_k \triangleq (E', \mathcal{I}')$

N : partition matroid

$I \in \mathcal{I}(M') \cap \mathcal{I}(N) \Leftrightarrow I \in \mathcal{I}(\tilde{M})$

$\max \{ |I| : I \in \mathcal{I}(\tilde{M}) \}$

$= \min_{T \subseteq E'} \{ r_1(T \cap E_1) + \dots + r_k(T \cap E_k) + |\{e \in E : e' \in T \wedge e' \notin \phi\}| \}$

EX: $M_1 \vee \dots \vee M_k$ is a matroid

$= \min_{T \subseteq E} \{ r_1(T \cap E_1) + \dots + r_k(T \cap E_k) + |E \setminus T| \}$

Cor: M^{k*} has max size of ind set $= \min_{T \subseteq E} \{ k \cdot r(I) + |E \setminus T| \}$

Cor: (Covering): min # ind set to cover $E = \max_{T \subseteq E} \left\lceil \frac{|T|}{r(I)} \right\rceil$

$\exists k$ ind set covers $E \Leftrightarrow k \cdot r(T) + |E \setminus T| \geq |E| \quad \forall T \subseteq E$

$$\Leftrightarrow k \geq \max_{T \subseteq E} \frac{|T|}{r(T)}$$

Cor: (Packing): max # disjoint bases = $\min_{T \subseteq E} \left\lfloor \frac{|E \setminus T|}{r(E) - r(T)} \right\rfloor$

$\exists k$ disjoint bases $\Leftrightarrow k \cdot r(T) + |E \setminus T| \geq k \cdot r(E) \quad \forall T \subseteq E$

$$\Leftrightarrow k \leq \min_{T \subseteq E} \frac{|E \setminus T|}{r(E) - r(T)}$$

Thm (Tutte, Nash-Williams): Arbitrarily: $\max_{U \subseteq V} \left\lfloor \frac{|E(U)|}{|U| - 1} \right\rfloor$

" $\Rightarrow$ " $B \in \mathcal{B}(M_1) \cap \mathcal{B}(M_2) \Leftrightarrow E \in \mathcal{I}(M_1, M_2^*)$

" $\Leftarrow$ " $E = E_1 \cup E_2$, s.t. $E_1 \in \mathcal{I}(M_1)$, $E_2 \in \mathcal{I}(M_2^*)$

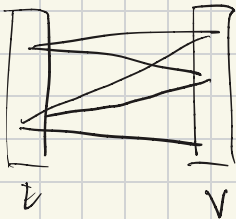
$$|E_1| \leq r, \quad |E_2| \leq r(M_2^*) = n - r$$

$$|E_1| + |E_2| = n \Rightarrow |E_1| = r, \quad |E_2| = n - r \Rightarrow E_1 \in \mathcal{B}(M_1) \cap \mathcal{B}(M_2)$$

$I \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)$, $|I| = k \Rightarrow$ take M_1/k , M_2/k .

eg: Transversal matroid:

sets elements



$s \in I \text{ if } |N(T)| \geq |T| \quad \forall T \subseteq s$

$$r(T) \leq |N(T)| \quad \forall T \subseteq E$$

Construction of matroid from polymatroid.

E' $f(e)$ copies of e
 $P(e)$

Q: why is this a matroid?

for $I' \subseteq E'$, $I' \in I$ if $\sum_{e \in S} |I' \cap P(e)| \leq f(S) \quad \forall S \subseteq E$

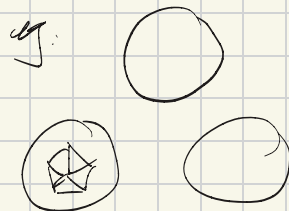
$$r(S) = \min_{T \subseteq S} \{ f(T) + |S \setminus T| \}$$

principal partition sequence.

$$f_\lambda(x) = r(x) - \lambda \cdot |x|$$

"increases" minimizers.

$$\bar{P}(\lambda) = \arg \min_{x \in E} f_\lambda(x)$$



continuously increase λ .

break points $\lambda_1 < \lambda_2 < \dots < \lambda_k$ when (unique) maximal minimizer changes.

obtain $\emptyset = P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_k = E$.

"principal partition sequence"

Lemma 1: $\lambda_1 = \min_{\emptyset \neq x \in E} \frac{r(x)}{|x|}$, P_1 is maximal in $\arg \min_{\emptyset \neq x \in E} \frac{r(x)}{|x|}$

$$\text{pf: } \min_{\emptyset \neq x \in E} \{r(x) - \lambda_1 |x|\} = 0 \Rightarrow r(x) \geq \lambda_1 |x| \quad \forall x \in E$$
$$\& \exists P_1, \text{ s.t. } r(P_1) = \lambda_1 |P_1|$$

Lemma: $\lambda_k = \max_{x \notin E} \frac{r(E) - r(x)}{|E| - |x|}$

$$\text{pf: } \min_{\emptyset \neq x \in E} \{r(x) - \lambda_k |x|\} = r(E) - \lambda_k |E|$$

$$\Rightarrow r(x) - \lambda_k |x| \geq r(E) - \lambda_k |E| \quad \forall x \in E$$

$$\text{v.e. } \lambda_k \geq \frac{r(E) - r(x)}{|E| - |x|} \quad \forall x \notin E$$

Moreover, equality holds for $P_{k-1} \subsetneq E$

Lemma 2: $\forall \lambda < \mu, X \in \mathcal{P}(\Lambda), Y \in \mathcal{P}(\Lambda) : X \subseteq Y$.

pf: Suppose not.

$$f_{\mu}(X) + f_{\mu}(Y) \geq f_{\mu}(X \cup Y) + f_{\mu}(X \cap Y)$$

$$\begin{aligned} f_{\mu}(X) - f_{\mu}(X \cap Y) &= (r(X) - \mu|X|) - (r(X \cap Y) - \mu|X \cap Y|) \\ &< (r(X) - \lambda|X|) - (r(X \cap Y) - \lambda|X \cap Y|) \\ &= f_{\lambda}(X) - f_{\lambda}(X \cap Y) \leq 0 \end{aligned}$$

$$\Rightarrow f_{\mu}(Y) > f_{\mu}(X \cup Y) \quad \rightarrow \Leftarrow$$

$$\text{Lemma 2: } \lambda_i = \min_{X \subseteq E \setminus F_i} \frac{r(X|F_0)}{|X|} \Rightarrow \lambda_i = \frac{r(\bar{F}_i \cap F_0) - r(F_i)}{|F_0 \setminus F_i|}$$

pf: Apply lemma 1 to M/F_i

λ_i : density of part $F_0 \setminus F_i$

eg: Graphic matroids.

Poly matroid: $f: 2^E \rightarrow \mathbb{R}_{\geq 0}$ "diminishing return": $f(e|s) \leq f(e|t)$
 $\forall t \subseteq s$

1. (monotone)

2. (submodular)

3. (normalized): $f(\emptyset) = 0$

Def: The poly matroid associated with f is $P_f := \{x \in \mathbb{R}_{\geq 0}^E : x(e) \leq f(e) \forall e \in E\}$

Base polytope: $B_f := P_f \cap \{x : x(E) = f(E)\}$

Matroid: let f be a matroid rank function. $I \in \mathcal{I}$ iff $x(I) \in P_f \cap \mathbb{Z}^E$

Optimization over poly matroid: $\max_{x \in P_f} w^T x$

Greedy: Sort $w_1 \geq w_2 \geq \dots \geq w_n$

Let $x_i^* := f(\{i_1, \dots, i_i\}) - f(\{i_1, \dots, i_{i-1}\}) \Rightarrow x_i^*(e) = f(e)$

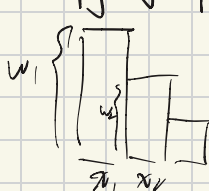
Pf of feasibility: $\forall S \subseteq E$ let $S_i := S \cap \{i_1, \dots, i_i\}$

$$x_i^*(S) = \sum_{e \in S} f(e | \{i_1, \dots, i_{i-1}\}) \leq \sum_{e \in S} f(e | S_{i-1}) = \sum_{e \in S} (f(S_{i-1} \cup e) - f(S_{i-1})) = f(S)$$

Cor: $B_f \subseteq \mathbb{Q}$

Pf of optimality: $\forall x \in P_f$:

$$w^T x = \sum_{i=1}^n (w_i - w_{i+1}) \cdot \left(\sum_{j=1}^i x_j \right) \leq \sum_{i=1}^n (w_i - w_{i+1}) \cdot f(\{i_1, \dots, i_i\})$$

$$= \sum_{i=1}^n (w_i - w_{i+1}) \cdot \left(\sum_{j=1}^i x_j^* \right) = w^T x^*$$


Submodular minimization:

Def (Convex envelope): $f \in \mathbb{R}^n \rightarrow \mathbb{R}$, the convex envelope is

$$\text{conv}(f)(x) = \sup \{ g(x) : g \text{ is convex}, g \leq f \} \quad (\text{outer description})$$

$$\text{Equivalently, } \text{conv}(f)(x) = \inf \left\{ \sum_{i=1}^{n-1} \lambda_i f(x^{(i)}) : \sum \lambda_i = 1, \lambda_i \geq 0 \right\} \quad (\text{inner description})$$

Lovász extension

$$w \in [0, 1]^n$$

$$(P): \max w^T x + b$$

$$\text{s.t. } x(S) + b \leq f(S) \quad \forall S \subseteq E$$

$$\text{conv}(f)(w) = \min \sum_{S \subseteq E} \lambda_S \cdot f(S)$$

$$\text{s.t. } \sum_{S \subseteq E} \lambda_S = 1$$

$$\lambda \geq 0$$

$$\sum_{S \subseteq E: i \in S} \lambda_S = w_i \quad \forall i \in [n]$$

$$b^* := f(\emptyset)$$

$$x_i^* := f(\{i\}) - f(\emptyset), \quad w_1 \geq \dots \geq w_n \geq w_{n+1} := 0$$

If f optimality:

$$w^T x^* + b^* = \sum_{i=1}^n (w_i - w_{i+1}) (f(\{1, \dots, i\}) - f(\emptyset)) + f(\emptyset)$$

$$= \sum_{i=1}^n (w_i - w_{i+1}) f(\{1, \dots, i\}) + (1 - w_1) \cdot f(\emptyset)$$

$$= \sum \lambda_S \cdot f(S), \quad \text{where}$$

$$\begin{array}{l} S \\ \emptyset \\ \{1\} \\ \{1, \dots, n\} \end{array} \quad \begin{array}{l} \lambda_S \\ 1 - w_1 \\ w_1 - w_2 \\ \vdots \\ w_n \end{array}$$

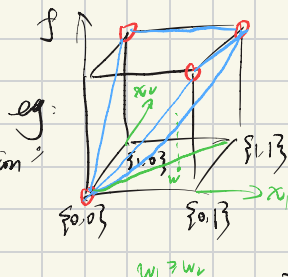
Then: f is submodular $\Leftrightarrow \text{conv}(f) = \hat{f}$

$$\text{where } \hat{f}(w) = \mathbb{E}(f(\{i: w_i \geq \theta\}))$$

$$\theta \sim U[0, 1]$$

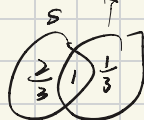
$\Rightarrow \checkmark$

"Kuhn triangulation"



$\Leftarrow$ For any $s, T \subseteq E$, let

$$w = \frac{1}{3}(\mathbb{1}_{s \cap T} + \mathbb{1}_s + \mathbb{1}_{s \cup T}) = \frac{1}{3} \cdot \mathbb{1}_T + \frac{2}{3} \cdot \mathbb{1}_s$$



$$\hat{f}(w) = \frac{1}{3} (f(s \cap T) + f(s) + f(s \cup T)) \leq \frac{1}{3} f(T) + \frac{2}{3} f(s)$$

$$\Rightarrow f(s \cap T) + f(s \cup T) \leq f(s) + f(T)$$

Ellipsoid alg for submodular minimization.

Separation $\Leftrightarrow$ optimization.

$$\text{Lemma: } \min\{f(x)\} = \min\{\text{conv}(f(x))\}$$

$$\text{Separation oracle for } \text{conv}(\text{epi}(f)) = \text{epi}(\text{conv}(f)) = \{t, w : t \geq \hat{f}(w)\}$$

Given (t_0, w_0)

$$\text{compute } \hat{f}(w_0) \leftarrow \begin{cases} \text{if } t_0 \geq \hat{f}(w_0) & \text{feasible} \\ \text{else} & \end{cases}$$

compute (x^*, b^*) , return separating hyperplane $t \geq (x^*)^T w + b^*$

Cor (of optimality of greedy): If $f: 2^V \rightarrow \mathbb{Z}_{\geq 0}$, P_f is integral &

TDI

$$\text{Alternative } f: \quad \chi(s) = f(s), \quad \chi(T) = f(T)$$

$$f(s \cap T) + f(s \cup T) \geq \chi(s \cap T) + \chi(s \cup T) = \chi(s) + \chi(T) = f(s) + f(T) \geq f(s \cap T) + f(s \cup T)$$

$$\Rightarrow \chi(s \cap T) = f(s \cap T), \quad \chi(s \cup T) = f(s \cup T) \quad \text{"uncross"}$$

$\Rightarrow$ independent tight constraints forming a "chain" $s_1 \subseteq s_2 \subseteq \dots \subseteq s_n$

$$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 1 & 1 \\ - & - & - & - \end{matrix}$$

totally unimodular

Ghouila-Houri

(poly) matroid intersection: $(f_1(\phi)=0)$

$$\max w^T x$$

$$\text{s.t. } \left. \begin{array}{l} x(s) \in f_1(s) \\ x(s) \in f_2(s) \\ x \geq 0 \end{array} \right\} (*)$$

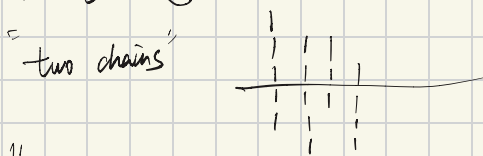
$$\min \sum \alpha_s f_1(s) + \sum \beta_s f_2(s)$$

$$\text{s.t. } \sum_{s: i \in s} \alpha_s + \sum_{s: i \in s} \beta_s \geq w_i$$

$$\alpha, \beta \geq 0$$

Then (weight splitting): $\max \{w^T x : x \in P_1 \cap P_2\} = \min \{ \hat{f}_1(w_1) + \hat{f}_2(w_2) : w_1 + w_2 = w \}$

Then: (*) is integral, and in fact TDI



||
weighted matroid intersection.