Matroids and Submodular Optimization September 2, 2026

Lecture 4

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Convention. The ground set E is finite, with $|E| = n$. For $x \in \mathbb{R}^E$ and $S \subseteq E$, write $x(S) = \sum_{e \in S} x_e$. For a set function f and $e \notin S$, write

$$f(e \mid S) = f(S + e) - f(S).$$

The incidence vector of S is $\mathbf{1}_S$.

1 Polymatroids

Definition 1.1 (Polymatroid rank function)**.** A function $f : 2^E \to \mathbb{R}_{\geq 0}$ is a *polymatroid rank function* if it is

(i) normalized: $f(\varnothing) = 0$;

(ii) nondecreasing: $f(S) \leq f(T)$ whenever $S \subseteq T$; and

(iii) submodular:
$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T) \qquad (S, T \subseteq E).$$

Equivalently, submodularity can be written in diminishing-returns form:

$$f(e \mid S) \leq f(e \mid T) \qquad \text{whenever } T \subseteq S \subseteq E \setminus \{e\}. \tag{1}$$

Definition 1.2 (Polymatroid and base polyhedra)**.** The polymatroid associated with f is

$$P(f) = \{x \in \mathbb{R}^E_{\geq 0} : x(S) \leq f(S) \text{ for every } S \subseteq E\}. \tag{2}$$

Its base polytope is the face

$$B(f) = \{x \in P(f) : x(E) = f(E)\}. \tag{3}$$

If f is integer-valued, $P(f)$ is called an *integral polymatroid*.

Example 1.3 (Matroid independence polytope)**.** Let $M = (E, \mathcal{I})$ have rank function r. Then r is a polymatroid rank function, and

$$P(r) \cap \mathbb{Z}^E = \{\mathbf{1}_I : I \in \mathcal{I}\}.$$

Indeed, the singleton inequalities force an integral point of $P(r)$ to be zero–one, and $\mathbf{1}_I \in P(r)$ holds exactly when I is independent. Thus the matroid independence polytope is a special case of a polymatroid.

2 Greedy optimization over a polymatroid

Consider

$$\max\{w^{\mathsf{T}} x : x \in P(f)\}, \qquad w \in \mathbb{R}^E_{\geq 0}. \tag{4}$$

Choose an ordering $\pi(1), \ldots, \pi(n)$ for which

$$w_{\pi(1)} \geq w_{\pi(2)} \geq \cdots \geq w_{\pi(n)} \geq 0,$$

and put

$$S_i = \{\pi(1), \ldots, \pi(i)\}, \qquad S_0 = \varnothing.$$

Greedy algorithm. Set

$$x^*_{\pi(i)} = f(S_i) - f(S_{i-1}) = f(\pi(i) \mid S_{i-1}) \qquad (i = 1, \ldots, n). \tag{5}$$

Theorem 2.1 (Polymatroid greedy theorem)**.** *The vector x^* in* (5) *belongs to $B(f)$ and is an optimal solution of* (4)*. In particular, $B(f)$ is nonempty.*

Proof. Monotonicity gives $x^* \geq 0$, and the definition telescopes to $x^*(E) = f(E)$. It remains to prove the inequalities in (2). Fix $A \subseteq E$ and let $A_i = A \cap S_i$. By diminishing returns,

$$\begin{aligned} x^*(A) &= \sum_{\pi(i) \in A} f(\pi(i) \mid S_{i-1}) \\ &\leq \sum_{\pi(i) \in A} f(\pi(i) \mid A_{i-1}) \\ &= \sum_{\pi(i) \in A} \big(f(A_i) - f(A_{i-1})\big) = f(A). \end{aligned}$$

Thus $x^* \in B(f)$.

For optimality, set $w_{\pi(n+1)} = 0$. Summation by parts gives, for every $x \in P(f)$,

$$\begin{aligned} w^{\mathsf{T}} x &= \sum_{i=1}^{n} \big(w_{\pi(i)} - w_{\pi(i+1)}\big) x(S_i) \\ &\leq \sum_{i=1}^{n} \big(w_{\pi(i)} - w_{\pi(i+1)}\big) f(S_i) = w^{\mathsf{T}} x^*. \end{aligned}$$

All coefficients in the inequality are nonnegative, so the comparison is valid. □

Remark (The role of the weight signs)*.* Nonnegativity of w is necessary in the displayed theorem over $P(f)$: a negative-weight coordinate should be left at zero. For arbitrary w, sort all coordinates and apply the same construction only to the positive-weight prefix. Over the base polytope $B(f)$, the full greedy vector (5) is optimal for arbitrary real weights; the last term in summation by parts is then $w_{\pi(n)} x(E) = w_{\pi(n)} f(E)$.

3 Convex envelopes and the Lovász extension

For the rest of this section, let $f : 2^E \to \mathbb{R}$ be an arbitrary set function. It need not be normalized, nondecreasing, or nonnegative. We identify a set S with the vertex $\mathbf{1}_S$ of the cube $[0,1]^E$.

Definition 3.1 (Convex extension)**.** A function $F : [0,1]^E \to \mathbb{R}$ is an *extension* of f if

$$F(\mathbf{1}_S) = f(S) \qquad (S \subseteq E).$$

It is a *convex extension* if it is also convex.

Definition 3.2 (Convex envelope or convex closure)**.** The convex envelope of f is the pointwise largest convex function lying below the prescribed values at the vertices. Its outer description is

$$(\operatorname{conv} f)(z) = \sup\{g(z) : g \text{ is convex and } g(\mathbf{1}_S) \le f(S) \text{ for all } S \subseteq E\}.$$

Equivalently, its inner description is

$$(\operatorname{conv} f)(z) = \min\left\{ \sum_{S\subseteq E} \lambda_S f(S) : \begin{array}{l} \sum_S \lambda_S \mathbf{1}_S = z, \\ \sum_S \lambda_S = 1, \quad \lambda_S \ge 0 \end{array} \right\}. \tag{6}$$

Because a cube vertex has no nontrivial convex representation by other cube vertices, $(\operatorname{conv} f)(\mathbf{1}_S) = f(S)$. Thus the convex envelope is itself a convex extension. By Carathéodory's theorem, the minimum in (6) has a solution supported on at most $n+1$ sets.

Writing $a(S) = \sum_{e\in S} a_e$, the LP dual of (6) is

$$(\operatorname{conv} f)(z) = \max\{a^\mathsf{T} z + b : a(S) + b \le f(S) \text{ for all } S \subseteq E\}. \tag{7}$$

Definition 3.3 (Lovász extension)**.** Let $z \in [0,1]^E$, and choose an order π such that

$$z_{\pi(1)} \ge \cdots \ge z_{\pi(n)}, \qquad z_{\pi(n+1)} = 0.$$

With $S_i = \{\pi(1), \dots, \pi(i)\}$, the *Lovász extension* of f is

$$\widehat{f}(z) = (1 - z_{\pi(1)}) f(\varnothing) + \sum_{i=1}^{n} (z_{\pi(i)} - z_{\pi(i+1)}) f(S_i). \tag{8}$$

Equivalently,

$$\widehat{f}(z) = f(\varnothing) + \sum_{i=1}^{n} z_{\pi(i)} \big(f(S_i) - f(S_{i-1})\big). \tag{9}$$

The definition is independent of how ties are ordered.

Threshold interpretation. If θ is uniform on $[0,1]$ and $S_\theta(z) = \{e \in E : z_e \ge \theta\}$, then

$$\widehat{f}(z) = \mathbb{E}_{\theta\sim\mathrm{Unif}[0,1]}\big[f(S_\theta(z))\big]. \tag{10}$$

Indeed, the threshold set equals S_i for an interval of length $z_{\pi(i)} - z_{\pi(i+1)}$, and it is empty on an interval of length $1 - z_{\pi(1)}$.

Theorem 3.4 (Submodularity and convexity). *For a set function $f : 2^E \to \mathbb{R}$, the following are equivalent:*

(i) f *is submodular;*

(ii) $\widehat{f} = \operatorname{conv} f$ *on* $[0,1]^E$*;*

(iii) the Lovász extension $\widehat{f}$ *is convex.*

Proof. Assume first that f is submodular. Fix z and its ordering π, and define

$$a_{\pi(i)} = f(S_i) - f(S_{i-1}), \qquad b = f(\varnothing).$$

For any $A \subseteq E$, the same diminishing-returns calculation as in the greedy proof gives

$$a(A) + b \le f(A).$$

Hence (a, b) is feasible for (7). On the other hand, the chain coefficients

$$\lambda_\varnothing = 1 - z_{\pi(1)}, \qquad \lambda_{S_i} = z_{\pi(i)} - z_{\pi(i+1)} \tag{11}$$

are feasible for (6). The two objectives agree:

$$a^\mathsf{T} z + b = (1 - z_{\pi(1)}) f(\varnothing) + \sum_{i=1}^{n} (z_{\pi(i)} - z_{\pi(i+1)}) f(S_i) = \widehat{f}(z).$$

Weak duality therefore proves $\operatorname{conv} f(z) = \widehat{f}(z)$. Since a convex envelope is convex, (i) implies both (ii) and (iii).

Conversely, the chain coefficients in (11) show for every set function that $\operatorname{conv} f \le \widehat{f}$. If $\widehat{f}$ is convex, then it is a convex extension of f, so the defining maximality of $\operatorname{conv} f$ gives $\widehat{f} \le \operatorname{conv} f$. Thus (iii) implies (ii).

It remains to prove that (ii) implies submodularity, following the one-third construction. For $S, T \subseteq E$, put

$$z = \frac{2}{3}\mathbf{1}_S + \frac{1}{3}\mathbf{1}_T = \frac{1}{3}(\mathbf{1}_{S\cap T} + \mathbf{1}_S + \mathbf{1}_{S\cup T}).$$

The threshold formula gives

$$\widehat{f}(z) = \frac{1}{3}(f(S \cap T) + f(S) + f(S \cup T)),$$

whereas the first convex representation of z gives

$$(\operatorname{conv} f)(z) \le \frac{2}{3} f(S) + \frac{1}{3} f(T).$$

Under (ii) these two expressions imply

$$f(S \cap T) + f(S \cup T) \le f(S) + f(T),$$

which is submodularity. $\square$

4 Submodular-function minimization

Lemma 4.1 (Discrete and continuous minima)**.** *If f is submodular, then*

$$\min_{S\subseteq E} f(S) = \min_{z\in[0,1]^E} \widehat{f}(z). \tag{12}$$

Proof. By (10), $\widehat{f}(z)$ is an average of values of f, so it is at least $\min_S f(S)$. Conversely, every set S is represented by $z = \mathbf{1}_S$, where $\widehat{f}(\mathbf{1}_S) = f(S)$. □

The right-hand side of (12) is a convex optimization problem by theorem 3.4. The required separation oracle comes from the same greedy chain.

Proposition 4.2 (Epigraph separation oracle)**.** *Suppose f is given by a value oracle. A separation oracle for*

$$\mathrm{epi}(\widehat{f}) = \{(t,z) \in \mathbb{R}\times[0,1]^E : t \geq \widehat{f}(z)\}$$

uses a sort and at most $n+1$ evaluations of f.

Proof. First separate a point outside the cube by a box inequality. Given (t_0, z^0) with $z^0 \in [0,1]^E$, sort the coordinates and form the chain $S_0, \dots, S_n$. Set

$$a_{\pi(i)} = f(S_i) - f(S_{i-1}), \qquad b = f(\varnothing). \tag{13}$$

The proof of theorem 3.4 shows that

$$a^{\mathsf{T}} z + b \leq \widehat{f}(z) \quad \text{for every } z \in [0,1]^E, \qquad a^{\mathsf{T}} z^0 + b = \widehat{f}(z^0).$$

If $t_0 \geq \widehat{f}(z^0)$, the point is feasible. Otherwise the valid supporting inequality

$$t \geq a^{\mathsf{T}} z + b \tag{14}$$

separates (t_0, z^0) from the epigraph. □

Corollary 4.3 (Value-oracle submodular minimization)**.** *Under the usual rational encoding assumptions, the ellipsoid method applied to* (12) *and theorem 4.2 minimizes a submodular function in polynomial time.*

The separation–optimization equivalence and the ellipsoid method are used here as black boxes; the explicit oracle is the part particular to submodular functions.

5 Integrality and total dual integrality

In this section the displayed systems are assumed rational whenever TDI terminology is used. Integer-valued right-hand sides will be stated separately when they are needed for primal integrality.

Definition 5.1 (Total dual integrality)**.** For the nonnegative-variable form used here, a rational system $Ax \leq b$, $x \geq 0$, is *totally dual integral* (TDI) if, for every integral objective w for which the primal optimum is finite, the dual

$$\min\{b^{\mathsf{T}} y : A^{\mathsf{T}} y \geq w,\ y \geq 0\}$$

has an integral optimal solution.

Theorem 5.2 (The polymatroid system is TDI)**.** *For a rational-valued polymatroid rank function f, the system*

$$x(S) \le f(S) \quad (S \subseteq E), \qquad x \ge 0,$$

is TDI. If f is integer-valued, then $P(f)$ is an integral polytope.

Proof. Let $w \in \mathbb{Z}^E$. Coordinates with $w_e \le 0$ need no dual coverage, so order the positive coordinates as

$$w_{\pi(1)} \ge \cdots \ge w_{\pi(m)} > 0, \qquad w_{\pi(m+1)} = 0,$$

and put $S_i = \{\pi(1), \dots, \pi(i)\}$. In the dual

$$\min \left\{ \sum_{S \subseteq E} f(S) y_S : \sum_{S \ni e} y_S \ge w_e \ (e \in E), \ y \ge 0 \right\}, \tag{15}$$

set

$$y_{S_i} = w_{\pi(i)} - w_{\pi(i+1)} \quad (i = 1, \dots, m), \qquad y_S = 0 \text{ otherwise.}$$

This is an integral feasible dual solution. Its objective equals the greedy primal value by summation by parts, so it is optimal. The system is TDI. When f is integer-valued, its right-hand side is integral, and the standard TDI integrality theorem implies that $P(f)$ is integral. This also follows directly because greedy produces an integral optimizer for every objective. □

The uncrossing picture. If $x(S) = f(S)$ and $x(T) = f(T)$ at a feasible point, then

$$\begin{aligned} f(S \cap T) + f(S \cup T) &\ge x(S \cap T) + x(S \cup T) \\ &= x(S) + x(T) = f(S) + f(T) \\ &\ge f(S \cap T) + f(S \cup T). \end{aligned}$$

Every inequality is therefore an equality, so $S \cap T$ and $S \cup T$ are tight. At a vertex, adjoin the active nonnegativity equations $x_e = 0$ and delete their columns. An uncrossed basis of the remaining tight-set incidence vectors can be chosen along a chain. The resulting chain matrix has the consecutive-ones property and is totally unimodular. This is the geometric counterpart of the chain dual in the greedy proof.

6 Polymatroid intersection

Let $f_1, f_2 : 2^E \to \mathbb{R}_{\ge 0}$ be rational-valued polymatroid rank functions. For $w \in \mathbb{R}^E_{\ge 0}$, the polymatroid-intersection LP and its dual are

$$\begin{aligned} \max \quad & w^\mathsf{T} x & & \qquad \text{(P)} \\ \text{s.t.} \quad & x(S) \le f_1(S) & & (S \subseteq E), \\ & x(S) \le f_2(S) & & (S \subseteq E), \\ & x \ge 0, & & \end{aligned}$$

and

$$\begin{aligned}
\min \quad & \sum_{S\subseteq E} \alpha_S f_1(S) + \sum_{S\subseteq E} \beta_S f_2(S) && \text{(D)}\\
\text{s.t.} \quad & \sum_{S\ni e} \alpha_S + \sum_{S\ni e} \beta_S \geq w_e && (e\in E),\\
& \alpha, \beta \geq 0.
\end{aligned}$$

For a normalized, monotone f, extend the Lovász extension positively homogeneously to $u \in \mathbb{R}^E_{\geq 0}$ by

$$\widehat{f}(u) = \sum_{i=1}^{n} (u_{\pi(i)} - u_{\pi(i+1)}) f(S_i), \qquad u_{\pi(n+1)} = 0. \tag{16}$$

The greedy theorem says precisely that

$$\widehat{f}(u) = \max\{u^{\mathsf{T}} x : x \in P(f)\}. \tag{17}$$

Theorem 6.1 (Weight splitting)**.** *For $w \in \mathbb{R}^E_{\geq 0}$,*

$$\max_{x\in P(f_1)\cap P(f_2)} w^{\mathsf{T}} x = \min_{\substack{w^1,w^2\in\mathbb{R}^E_{\geq 0}\\ w^1+w^2=w}} (\widehat{f}_1(w^1) + \widehat{f}_2(w^2)). \tag{18}$$

The proof is an application of LP duality: the two dual families split the weight between the support functions of $P(f_1)$ and $P(f_2)$. It is omitted here, as in the source notes.

Theorem 6.2 (Polymatroid-intersection TDI theorem)**.** *The system in* (P) *is TDI. Consequently, if f_1 and f_2 are integer-valued, then $P(f_1) \cap P(f_2)$ is an integral polytope.*

Proof sketch: two chains. Choose an optimal solution (α, β) of (D), and among all optimal solutions choose one minimizing

$$\Phi(\alpha, \beta) = \sum_S \alpha_S |S||E \setminus S| + \sum_S \beta_S |S||E \setminus S|.$$

Suppose incomparable sets A, B have positive α-coefficients. For $\varepsilon = \min\{\alpha_A, \alpha_B\}$, move ε units from A, B to $A \cap B, A \cup B$. The identity

$$\mathbf{1}_A + \mathbf{1}_B = \mathbf{1}_{A\cap B} + \mathbf{1}_{A\cup B}$$

preserves every dual covering constraint, while submodularity of f_1 does not increase the objective. Optimality makes the objective unchanged, but

$$|A||E \setminus A| + |B||E \setminus B| - |A \cap B||E \setminus (A \cap B)| - |A \cup B||E \setminus (A \cup B)| = 2|A \setminus B|\,|B \setminus A| > 0,$$

contradicting the choice of Φ. Thus $\operatorname{supp}(\alpha)$ is a chain; the same argument makes $\operatorname{supp}(\beta)$ a chain. It remains to examine the incidence matrix of these two chains. Order the sets of the first chain increasingly and those of the second chain decreasingly. In every element-column, the ones form a single consecutive interval. The matrix is therefore totally unimodular. Restricting the dual to these two chains preserves its optimum, and total unimodularity gives an integral optimal dual whenever w is integral. This is exactly TDI. With integer-valued f_1, f_2, TDI and the integral right-hand side imply primal integrality. $\square$

Corollary 6.3 (Weighted matroid intersection)**.** *Let $f_i = r_i$ be the rank functions of matroids $M_i = (E, \mathcal{I}_i)$. The integral points of $P(r_1) \cap P(r_2)$ are exactly*

$$\{\mathbf{1}_I : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}.$$

Consequently, (P) *is an exact LP formulation of weighted matroid intersection. Negative weights may be discarded because the feasible region is down-closed.*

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