Matroids and Submodular Optimization August 31, 2026

Lecture 3

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1 Matroid union

Definition 1.1 (Matroid union)**.** Let $M_i = (E_i, \mathcal{I}_i)$ for $i = 1, \dots, k$, and put $E = \bigcup_{i=1}^k E_i$. Define

$$\mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k = \{I_1 \cup \cdots \cup I_k : I_i \in \mathcal{I}_i \text{ for } i = 1, \dots, k\}.$$

The union is

$$M_1 \vee \cdots \vee M_k := (E, \mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k).$$

By heredity, repeated elements can be removed from the I_i. Thus a set is union-independent precisely when it can be *partitioned* into sets $I_1, \dots, I_k$ with $I_i \in \mathcal{I}_i$.

Theorem 1.2 (Matroid-union theorem)**.** *The system in theorem 1.1 is a matroid. For every $S \subseteq E$, its rank is*

$$r_{M_1 \vee \cdots \vee M_k}(S) = \min_{T \subseteq S} \left\{ |S \setminus T| + \sum_{i=1}^k r_i(T \cap E_i) \right\}. \tag{1}$$

1.1 Reduction to matroid intersection

Make labeled copies of the ground sets:

$$\widehat{E} = \biguplus_{i=1}^k (E_i \times \{i\}).$$

On row $E_i \times \{i\}$ place a copy $M_i^{(i)}$ of M_i, and form the direct sum

$$\widehat{M} = M_1^{(1)} \oplus \cdots \oplus M_k^{(k)}.$$

For $e \in E$, let

$$P_e = \{(e, i) : e \in E_i\},$$

and let N be the partition matroid with parts P_e, each of capacity one.

A common independent set of $\widehat{M}$ and N chooses at most one labeled copy of each original element. Projecting (e, i) to e gives a union-independent set of the same size. Conversely, partition a union-independent set among the M_i and lift each element to its assigned row. Therefore

$$\max\{|I| : I \in \mathcal{I}(M_1 \vee \cdots \vee M_k),\ I \subseteq S\} = \max\{|J| : J \in \mathcal{I}(\widehat{M}) \cap \mathcal{I}(N),\ J \subseteq \widehat{S}\},$$

where $\widehat{S} = \biguplus_i ((S \cap E_i) \times \{i\})$.

The matroid-intersection min–max theorem gives

$$\min_{A \subseteq \widehat{S}} \big(r_{\widehat{M}}(A) + r_N(\widehat{S} \setminus A) \big).$$

For any A, let T be the set of elements whose entire fibers lie in A. Then

$$r_{\widehat{M}}(A) \geq \sum_{i=1}^{k} r_i(T \cap E_i), \qquad r_N(\widehat{S} \setminus A) = |S \setminus T|.$$

Conversely, taking A to be the union of all fibers over T gives equality. This proves the rank formula (1).

For completeness, the right-hand side of (1) is submodular. Indeed, put $g(T) = \sum_i r_i(T \cap E_i)$, and let $X \subseteq A$ and $Y \subseteq B$ minimize the formula for A and B. Submodularity of g, together with

$$|A \setminus X| + |B \setminus Y| = |(A \cup B) \setminus (X \cup Y)| + |(A \cap B) \setminus (X \cap Y)|,$$

shows

$$r_\vee(A) + r_\vee(B) \geq r_\vee(A \cup B) + r_\vee(A \cap B).$$

The resulting integer-valued function is normalized, nondecreasing, and at most $|S|$; hence it is a matroid rank function. This also proves the first assertion of theorem 1.2.

1.2 The k-fold union of one matroid

For $M = (E, \mathcal{I})$, write

$$M^{\vee k} = \underbrace{M \vee \cdots \vee M}_{k \text{ copies}}.$$

Its independent sets are exactly the sets whose elements can be colored with k colors so that every color class is independent in M. Formula (1) becomes

$$r_{M^{\vee k}}(S) = \min_{T \subseteq S} \{|S \setminus T| + kr(T)\}. \tag{2}$$

Union versus direct sum. $M^{\vee k}$ has the original ground set E and asks for a coloring of its elements. The direct sum $M \oplus \cdots \oplus M$ has k disjoint labeled copies of E. It appears only in the copied-ground-set reduction.

2 Covering and packing consequences

2.1 Covering by independent sets

Definition 2.1 (Matroid chromatic number)**.** The matroid chromatic number $\chi(M)$ is the minimum number of independent sets whose union is E. Such a cover can always be made into a partition by deleting repeated elements.

Theorem 2.2 (Matroid covering theorem)**.** *If M is loopless, then*

$$\chi(M) = \max_{\varnothing \neq T \subseteq E} \left\lceil \frac{|T|}{r(T)} \right\rceil. \tag{3}$$

Equivalently, E can be covered by k independent sets if and only if

$$|T| \leq kr(T) \qquad \forall T \subseteq E.$$

If M has a loop, no collection of independent sets covers E.

Proof. The ground set is coverable by k independent sets exactly when $E \in \mathcal{I}(M^{\vee k})$, or equivalently when $r_{M^{\vee k}}(E) = |E|$. By (2), this holds exactly when

$$|E \setminus T| + kr(T) \geq |E| \qquad \forall T \subseteq E,$$

which is the displayed condition. Minimizing the integer k gives (3). □

Corollary 2.3 (Nash–Williams' arboricity theorem)**.** *For a loopless graph $G = (V, E)$, the minimum number of forests whose union is E is*

$$a(G) = \max_{\substack{U \subseteq V \\ |U| \geq 2}} \left\lceil \frac{|E(U)|}{|U| - 1} \right\rceil, \tag{4}$$

where $E(U)$ consists of the edges with both endpoints in U.

2.2 Packing disjoint bases

Theorem 2.4 (Matroid base-packing theorem)**.** *Let $r(E) > 0$. The matroid contains k pairwise disjoint bases if and only if*

$$|E \setminus T| \geq k\big(r(E) - r(T)\big) \qquad \forall T \subseteq E. \tag{5}$$

Consequently, the maximum number of pairwise disjoint bases equals

$$\min_{\substack{T \subseteq E \\ r(T) < r(E)}} \left\lfloor \frac{|E \setminus T|}{r(E) - r(T)} \right\rfloor. \tag{6}$$

Proof. There are k disjoint bases exactly when $M^{\vee k}$ has an independent set of size $kr(E)$. Since no union of k independent sets has more than $kr(E)$ elements, this is equivalent to $r_{M^{\vee k}}(E) = kr(E)$. Formula (2) gives

$$|E \setminus T| + kr(T) \geq kr(E) \qquad \forall T \subseteq E,$$

which is (5). Maximizing the integer k gives (6). □

For a connected graph, bases of the graphic matroid are spanning trees.

Theorem 2.5 (Tutte–Nash–Williams spanning-tree packing)**.** *The maximum number of pairwise edge-disjoint spanning trees in a connected graph G is*

$$\tau(G) = \min_{\substack{\mathcal{P} \text{ a partition of } V \\ |\mathcal{P}| \geq 2}} \left\lfloor \frac{|E(\mathcal{P})|}{|\mathcal{P}| - 1} \right\rfloor, \tag{7}$$

where $E(\mathcal{P})$ denotes the edges joining different parts of $\mathcal{P}$.

Remark (Attribution)*.* Equation (4) is Nash–Williams' arboricity theorem. Equation (7) is the Tutte–Nash–Williams spanning-tree-packing theorem. They are respectively the graphic specializations of matroid independent-set covering and matroid base packing.

2.3 From intersection back to union

Suppose M_1 and M_2 are matroids of the same rank r on an n-element ground set E. Then

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) \neq \varnothing \quad \Longleftrightarrow \quad E \in \mathcal{I}(M_1 \vee M_2^*). \tag{8}$$

Indeed, a common base B gives the decomposition $E = B \mathbin{\dot\cup} (E \setminus B)$, where B is independent in M_1 and $E \setminus B$ is a base of M_2^*. Conversely, suppose $E = I_1 \cup I_2$, with $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2^*)$. Since

$$|I_1| \leq r, \qquad |I_2| \leq n - r, \qquad |I_1 \cup I_2| = n,$$

equality holds throughout. Thus I_1 and I_2 are disjoint, $I_1 = E \setminus I_2$, and I_1 is a base of both M_1 and M_2.

To ask for a common independent set of prescribed size q, truncate both matroids to rank q. Writing $M_i^{(q)}$ for the rank-q truncation,

$$\exists I \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2),\ |I| = q \quad \Longleftrightarrow \quad E \in \mathcal{I}\big(M_1^{(q)} \vee (M_2^{(q)})^*\big),$$

provided $q \leq \min\{r_1(E), r_2(E)\}$.

3 The principal partition sequence

Let $M = (E, \mathcal{I})$ have nonempty ground set and rank function r. For $\lambda \in \mathbb{R}$, define the parametric submodular function

$$h_\lambda(X) = r(X) - \lambda |X|, \tag{9}$$

its family of minimizers

$$\mathcal{F}(\lambda) = \arg\min_{X \subseteq E} h_\lambda(X),$$

and its largest minimizer

$$F(\lambda) = \bigcup_{X \in \mathcal{F}(\lambda)} X. \tag{10}$$

Lemma 3.1 (Lattice of minimizers)**.** *For every λ, the family $\mathcal{F}(\lambda)$ is closed under union and intersection. Consequently, $F(\lambda)$ is the unique largest minimizer; there is likewise a unique smallest minimizer.*

Proof. The function h_λ is submodular. If A, B are minimizers with common minimum value m, then

$$2m = h_\lambda(A) + h_\lambda(B) \geq h_\lambda(A \cap B) + h_\lambda(A \cup B) \geq 2m.$$

Both inequalities are equalities, so $A \cap B$ and $A \cup B$ are minimizers. Since the ground set is finite, the union and intersection of all minimizers are respectively the largest and smallest minimizers. □

Lemma 3.2 (Parametric nestedness)**.** *If $\lambda < \mu$, $X \in \mathcal{F}(\lambda)$, and $Y \in \mathcal{F}(\mu)$, then $X \subseteq Y$.*

Proof. Optimality of X, compared with $X \cap Y$, gives

$$r(X) - r(X \cap Y) \leq \lambda |X \setminus Y|. \tag{11}$$

Optimality of Y, compared with $X \cup Y$, gives

$$r(X \cup Y) - r(Y) \geq \mu |X \setminus Y|. \tag{12}$$

Rank submodularity implies

$$r(X) - r(X \cap Y) \geq r(X \cup Y) - r(Y).$$

Combining this with (11) and (12),

$$\lambda |X \setminus Y| \geq r(X) - r(X \cap Y) \geq r(X \cup Y) - r(Y) \geq \mu |X \setminus Y|.$$

Because $\lambda < \mu$, we must have $X \setminus Y = \varnothing$. □

3.1 Critical values and principal blocks

Because E is finite, $F(\lambda)$ assumes only finitely many values. Let

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k$$

be the values at which the largest minimizer changes. With the maximal- minimizer convention, define $F_i = F(\lambda_i)$ and let $F_0 = \varnothing$. By theorem 3.2,

$$\varnothing = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E. \tag{13}$$

Definition 3.3 (Principal sequence and principal partition)**.** The chain (13) is the *principal sequence* of M. Its differences

$$P_i = F_i \setminus F_{i-1}, \qquad i = 1, \dots, k,$$

are the blocks of the *principal partition* of E.

At the breakpoint λ_i, the smallest minimizer is F_{i-1} and the largest minimizer is F_i. Other minimizers may lie between them; choosing the largest minimizer makes the sequence canonical.

3.2 The first and last breakpoints

Lemma 3.4 (First breakpoint)**.** *If $E \neq \varnothing$, then*

$$\lambda_1 = \min_{\varnothing \neq X \subseteq E} \frac{r(X)}{|X|}, \tag{14}$$

and F_1 is the unique largest set attaining this minimum.

Proof. The empty set has $h_\lambda(\varnothing) = 0$. It remains the unique maximal minimizer while $r(X) - \lambda |X| > 0$ for every nonempty X. The first tie therefore occurs at the minimum ratio in (14). At λ_1 all objectives are nonnegative, and equality holds exactly for the ratio minimizers. Their union is F_1 by theorem 3.1. □

Lemma 3.5 (Last breakpoint)**.** *The last critical value is*

$$\lambda_k = \max_{X \subsetneq E} \frac{r(E) - r(X)}{|E \setminus X|}. \tag{15}$$

Equality is attained by $X = F_{k-1}$.

Proof. The set E is a minimizer of h_λ exactly when

$$r(E) - \lambda|E| \le r(X) - \lambda|X| \qquad \forall X \subsetneq E,$$

or equivalently when λ is at least every ratio in (15). At the first value for which E becomes a minimizer, it ties with the preceding maximal minimizer F_{k-1}. □

3.3 The recursive density formula

For $F \subseteq E$ and $X \subseteq E \setminus F$, write

$$r(X \mid F) = r(F \cup X) - r(F).$$

This is the rank of X in the contraction M/F.

Proposition 3.6 (Recursive construction)**.** *For $i = 0, \dots, k-1$,*

$$\lambda_{i+1} = \min_{\varnothing \ne X \subseteq E \setminus F_i} \frac{r(X \mid F_i)}{|X|}, \tag{16}$$

and $P_{i+1} = F_{i+1} \setminus F_i$ is the largest minimizer. In particular,

$$\lambda_{i+1} = \frac{r(F_{i+1}) - r(F_i)}{|F_{i+1} \setminus F_i|}. \tag{17}$$

Proof. Apply theorem 3.4 to the contracted matroid M/F_i. Its rank function is $X \mapsto r(X \mid F_i)$, and its first principal block lifts to $F_{i+1} \setminus F_i$ in M. Formula (17) follows by taking $X = F_{i+1} \setminus F_i$ in (16). □

Definition 3.7 (Principal density)**.** The density of the block P_i is

$$\rho_i = \frac{|P_i|}{r(F_i) - r(F_{i-1})} = \frac{1}{\lambda_i}. \tag{18}$$

Consequently,

$$\rho_1 > \rho_2 > \cdots > \rho_k.$$

If P_1 consists of loops, then $\lambda_1 = 0$ and $\rho_1 = +\infty$.

With the convention $a/0 = +\infty$ for $a > 0$, the principal minor

$$M_i = (M|F_i)/F_{i-1}$$

is uniformly dense: for every nonempty $X \subseteq P_i$,

$$\frac{|X|}{r_{M_i}(X)} \le \rho_i,$$

with equality for $X = P_i$. Thus the principal partition decomposes a matroid into minors of successively decreasing density.

3.4 Covering and packing as the two extreme densities

For a loopless matroid, equations (3) and (14) give

$$\chi(M) = \lceil \rho_1 \rceil = \left\lceil \frac{1}{\lambda_1} \right\rceil. \tag{19}$$

If $r(E) > 0$, equations (6) and (15) give

$$\nu(M) = \lfloor \rho_k \rfloor = \left\lfloor \frac{1}{\lambda_k} \right\rfloor, \tag{20}$$

where $\nu(M)$ is the maximum number of disjoint bases. Thus the densest principal block controls covering, while the sparsest block controls base packing.

For a connected loopless graph G, these identities become

$$\rho_1 = \max_{\substack{U \subseteq V \\ |U| \geq 2}} \frac{|E(U)|}{|U|-1}, \qquad \rho_k = \min_{\substack{\mathcal{P} \text{ a partition of } V \\ |\mathcal{P}| \geq 2}} \frac{|E(\mathcal{P})|}{|\mathcal{P}|-1}.$$

The first is the fractional arboricity and the second is the graph strength. Taking a ceiling or floor recovers theorem 2.3 and the Tutte–Nash–Williams theorem.

Example 3.8 (A dense core and a sparse bridge)**.** Let G consist of a triangle on a, b, c and a bridge cd to a fourth vertex. In its graphic matroid,

$$F_1 = E(\{a, b, c\}), \qquad F_2 = E(G),$$

and the principal data are

$$\begin{array}{lll} P_1 & r(F_1) - r(F_0) = 2 & \rho_1 = 3/2, \\ P_2 & r(F_2) - r(F_1) = 1 & \rho_2 = 1. \end{array}$$

The triangle forces $a(G) = \lceil 3/2 \rceil = 2$, while the bridge forces $\tau(G) = \lfloor 1 \rfloor = 1$.

Algorithmic meaning. The entire principal sequence can be recovered by parametric minimization of $r(X) - \lambda|X|$; it has at most $|E|$ nonempty blocks. In the graphic case one can use specialized flow and forest routines. The sequence immediately returns the optimum values and canonical obstruction sets. Constructing the actual forest cover or the actual edge-disjoint spanning trees still requires a matroid-partition or packing algorithm.

References

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