Matroids and Submodular Optimization	August 26, 2026

Lecture 2

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1 The greedy algorithm

Let $w : E \to \mathbb{R}$ be an arbitrary weight function. To find a minimum-weight base, order the elements as

$$w(e_1) \le w(e_2) \le \cdots \le w(e_n),$$

breaking ties arbitrarily.

Greedy algorithm for a minimum-weight base. Initialize $A = \varnothing$. Scan $e_1, e_2, \ldots, e_n$ in this order. When $A + e_i \in \mathcal{I}$, replace A by $A + e_i$; otherwise discard e_i. Return A.

The output is a maximal independent set and hence a base. The nontrivial point is that it has minimum weight.

Definition 1.1 (Serial exchange)**.** Let A, B be bases of rank r, and fix an ordering $A = (a_1, \ldots, a_r)$. An ordering $B = (b_1, \ldots, b_r)$ is a *one-sided serial exchange from B to A along the prescribed order* if, with $A_i = \{a_1, \ldots, a_i\}$ and $B_0 = B$, one has

$$b_i \in B_{i-1} \setminus A_{i-1}, \qquad B_i = B_{i-1} - b_i + a_i \in \mathcal{B}(M) \qquad (i = 1, \ldots, r).$$

Equivalently,

$$B_i = B - \{b_1, \ldots, b_i\} + \{a_1, \ldots, a_i\} \in \mathcal{B}(M) \qquad (i = 0, \ldots, r).$$

The one-sided serial-exchange property says that every pair of bases admits such a sequence for every prescribed ordering of A. Moreover, the sequence may be chosen with $b_i = a_i$ whenever $a_i \in A \cap B$. We use this property without proof.

Theorem 1.2 (Rado–Edmonds greedy theorem)**.** *The greedy algorithm returns a minimum-weight base of M. Equivalently, scanning the elements in nonincreasing order returns a maximum-weight base.*

Proof. Let the greedy base be $A = \{a_1, \ldots, a_r\}$, listed in the order in which its elements were selected, and put $A_i = \{a_1, \ldots, a_i\}$. Fix any base O. By the one-sided serial-exchange property following theorem 1.1, order $O = (o_1, \ldots, o_r)$ so that

$$B_i = O - \{o_1, \ldots, o_i\} + \{a_1, \ldots, a_i\} \in \mathcal{B}(M) \qquad (i = 0, \ldots, r),$$

with $o_i \in B_{i-1} \setminus A_{i-1}$.

Because $A_{i-1} + o_i \subseteq B_{i-1}$, the set $A_{i-1} + o_i$ is independent. We claim that $w(a_i) \le w(o_i)$. If instead $w(o_i) < w(a_i)$, then greedy scanned o_i before a_i. At that earlier time its current set was a subset of A_{i-1}, so heredity and the independence of $A_{i-1} + o_i$ imply that o_i was feasible. Greedy would have accepted it, contrary to $o_i \notin A_{i-1}$. Summing the inequalities gives

$$w(A) = \sum_{i=1}^{r} w(a_i) \le \sum_{i=1}^{r} w(o_i) = w(O).$$

Since O was arbitrary, A is a minimum-weight base. $\square$

2 Stronger basis-exchange properties

Ordinary basis exchange changes one base while making no promise about the other. Several stronger statements are useful in algorithms and structural matroid theory.

Definition 2.1 (Exchange graph of a base)**.** For a base B, its *exchange graph* is the bipartite graph with parts B and $E \setminus B$, where $x \in B$ is adjacent to $y \notin B$ when $B - x + y$ is a base.

Theorem 2.2 (Symmetric basis exchange)**.** *Let B_1, B_2 be bases and let $x \in B_1 \setminus B_2$. There exists $y \in B_2 \setminus B_1$ such that both*

$$B_1 - x + y \qquad \textit{and} \qquad B_2 - y + x$$

are bases.

Theorem 2.3 (Brualdi's matching theorem)**.** *For any two bases B_1, B_2, there is a bijection $\varphi : B_1 \to B_2$, fixing every element of $B_1 \cap B_2$, such that*

$$B_1 - x + \varphi(x) \in \mathcal{B}(M) \qquad \forall x \in B_1.$$

Equivalently, the exchange graph between $B_1 \setminus B_2$ and $B_2 \setminus B_1$ contains a perfect matching.

The bijection in theorem 2.3 is one-sided: it does not necessarily make $B_2 - \varphi(x) + x$ a base for every x. Requiring all matched pairs to be symmetric is the substantially stronger base-orderability property.

Definition 2.4 (Serial symmetric exchange)**.** Let B_1, B_2 be bases, and let

$$X = \{x_1, \ldots, x_k\} \subseteq B_1 \setminus B_2, \qquad Y = \{y_1, \ldots, y_k\} \subseteq B_2 \setminus B_1.$$

The displayed orderings give a *serial symmetric exchange* if, for every $i = 1, \ldots, k$, both

$$B_1 - \{x_1, \ldots, x_i\} + \{y_1, \ldots, y_i\}$$

and

$$B_2 - \{y_1, \ldots, y_i\} + \{x_1, \ldots, x_i\}$$

are bases. Thus the pairs are exchanged one at a time, and both intermediate sets remain bases after every step.

Conjecture 2.5 (Gabow's serial-exchange conjecture)**.** *Any two disjoint bases admit a full serial symmetric exchange: their elements can be ordered*

$$B_1 = \{x_1, \ldots, x_r\}, \qquad B_2 = \{y_1, \ldots, y_r\}$$

so that every pair of prefix exchanges in the preceding definition consists of two bases.

3 Matroid intersection

Let

$$M_1 = (E, \mathcal{I}_1), \qquad M_2 = (E, \mathcal{I}_2)$$

be matroids on the same ground set. A set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is called a *common independent set.* In general, $\mathcal{I}_1 \cap \mathcal{I}_2$ is not itself a matroid.

Example 3.1 (Bipartite matching)**.** For a bipartite graph $G = (U, V; E)$, let M_U enforce degree at most one at each vertex of U, and let M_V enforce degree at most one at each vertex of V. Both are partition matroids, and their common independent sets are precisely the matchings of G.

Example 3.2 (Rooted arborescences)**.** For a directed graph with root s, intersect the graphic matroid of the underlying undirected graph with the partition matroid imposing indegree at most one at every vertex other than s and indegree zero at s. A common independent set of size $|V| - 1$ is an s-arborescence.

Theorem 3.3 (Edmonds' matroid-intersection theorem)**.** *For two matroids on the same finite ground set,*

$$\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min_{A \subseteq E}\{r_1(A) + r_2(E \setminus A)\}.$$

The easy direction is a rank bound. For every common independent set I and every $A \subseteq E$,

$$|I| = |I \cap A| + |I \cap (E \setminus A)| \le r_1(A) + r_2(E \setminus A). \tag{1}$$

The reverse inequality will follow from an augmenting-path algorithm that also constructs a set A attaining equality.

3.1 The exchange digraph

Fix a common independent set I. Its *matroid-intersection exchange digraph* $D(I)$ has vertex set E and bipartition $I, E \setminus I$. For $y \in I$ and $x \in E \setminus I$, include the arcs

$$\begin{aligned} x \longrightarrow y \quad &\iff \quad I - y + x \in \mathcal{I}_1, \\ y \longrightarrow x \quad &\iff \quad I - y + x \in \mathcal{I}_2. \end{aligned}$$

Define the source and target sets

$$S_I = \{x \in E \setminus I : I + x \in \mathcal{I}_2\}, \qquad T_I = \{x \in E \setminus I : I + x \in \mathcal{I}_1\}.$$

Remark (Orientation convention). This is the convention used throughout these notes: M_1-exchange arcs point into I, M_2-exchange arcs point out of I, and augmenting paths run from S_I to T_I. Some references reverse every arc; their paths are the reversals of ours.

A directed S_I-to-T_I path has the alternating form

$$P = (x_0, y_1, x_1, y_2, \ldots, y_k, x_k), \tag{2}$$

where the x_i lie outside I and the y_i lie in I. Augmentation toggles the vertices of P:

$$I' = I \triangle V(P) = I - \{y_1, \ldots, y_k\} + \{x_0, \ldots, x_k\}.$$

Thus $|I'| = |I| + 1$. Feasibility is subtler: an arbitrary alternating path need not work, which is why the algorithm uses a shortest one.

Unweighted matroid-intersection algorithm. Start with $I = \varnothing$. Construct $D(I)$ and find a shortest directed path from S_I to T_I, where shortest means fewest arcs. If a path P exists, replace I by $I \triangle V(P)$ and repeat. If no path exists, return I.

3.2 A unique-perfect-matching certificate

For one matroid $M = (E, \mathcal{I})$ and $I \in \mathcal{I}$, let $G_M(I)$ be the undirected bipartite graph between I and $E \setminus I$ in which

$$yx \in E(G_M(I)) \iff I - y + x \in \mathcal{I}.$$

Lemma 3.4 (Triangular form of a unique perfect matching)**.** *Let $G = (L, R; F)$ be a bipartite graph with $|L| = |R| = m$. The graph has a unique perfect matching if and only if its vertices can be ordered*

$$L = \{\ell_1, \dots, \ell_m\}, \qquad R = \{q_1, \dots, q_m\}$$

so that $\ell_i q_i$ are the matching edges and

$$N_G(\{\ell_1, \dots, \ell_j\}) = \{q_1, \dots, q_j\} \qquad (j = 1, \dots, m).$$

Proof. Fix a perfect matching and contract each matching edge to one vertex. Direct an unmatched edge $\ell_i q_j$ from the contracted pair i to the contracted pair j. A directed cycle is exactly an alternating cycle, and flipping a perfect matching around such a cycle produces a second perfect matching. Hence uniqueness is equivalent to acyclicity. A reverse topological ordering of the contracted digraph gives the displayed neighborhood property. Conversely, the property forces ℓ_1 to be matched to q_1; deleting this pair and applying induction forces the entire matching. □

Lemma 3.5 (Unique-perfect-matching lemma)**.** *Let $I \in \mathcal{I}$ and let $J \subseteq E$ satisfy $|J| = |I|$. If the graph*

$$G_M(I)[I \triangle J]$$

has a unique perfect matching between $I \setminus J$ and $J \setminus I$, then $J \in \mathcal{I}$.

Proof. Write $I \setminus J = \{y_1, \dots, y_m\}$ and $J \setminus I = \{x_1, \dots, x_m\}$ in the triangular order from the preceding lemma, with $y_i x_i$ in the unique perfect matching. Thus

$$y_i x_j \notin E(G_M(I)) \qquad \text{whenever } i < j. \tag{3}$$

Suppose for a contradiction that J is dependent, and choose a circuit $C \subseteq J$. Since $I \cap J$ is independent, C contains some element of $J \setminus I$; let i be the smallest index for which $x_i \in C$. Every element of $C - x_i$ lies in $\mathrm{cl}(I - y_i)$. Indeed, an element of $C - x_i$ that belongs to $I \cap J$ already lies in $I - y_i$. Every other such element is an x_j with $j > i$; by (3), $I - y_i + x_j$ is dependent, and hence $x_j \in \mathrm{cl}(I - y_i)$.

The circuit C gives $x_i \in \mathrm{cl}(C - x_i)$, so monotonicity and idempotence of closure imply

$$x_i \in \mathrm{cl}(C - x_i) \subseteq \mathrm{cl}(I - y_i).$$

This contradicts the matching edge $y_i x_i$, which says that $I - y_i + x_i$ is independent. Therefore J is independent. □

3.3 Why a shortest path augments

Lemma 3.6 (Shortest-path augmentation)**.** *If P is a shortest S_I-to-T_I path in $D(I)$, then*

$$I \triangle V(P) \in \mathcal{I}_1 \cap \mathcal{I}_2.$$

Proof. Write P as in (2). First consider

$$J_1 = I - \{y_1, \dots, y_k\} + \{x_0, \dots, x_{k-1}\}.$$

The M_1-arcs $x_{i-1} \to y_i$ form a perfect matching in $G_{M_1}(I)[I \triangle J_1]$. This matching is unique. Indeed, a different perfect matching would contain an edge $x_j y_i$ with $j < i - 1$, and the arc $x_j \to y_i$ would shortcut P. By theorem 3.5, $J_1 \in \mathcal{I}_1$.

None of $x_0, \dots, x_{k-1}$ belongs to T_I, since otherwise the prefix of P ending there would be shorter. Hence these elements lie in $\mathrm{cl}_1(I)$, so $J_1 \subseteq \mathrm{cl}_1(I)$. Since J_1 and I are independent and have the same size, $\mathrm{cl}_1(J_1) = \mathrm{cl}_1(I)$. The target property gives $x_k \notin \mathrm{cl}_1(I)$, and therefore

$$I \triangle V(P) = J_1 + x_k \in \mathcal{I}_1.$$

For M_2, use

$$J_2 = I - \{y_1, \dots, y_k\} + \{x_1, \dots, x_k\}.$$

The path arcs $y_i \to x_i$ give the unique perfect matching in the M_2 exchange graph; another matching would contain some $y_i x_j$ with $j > i$ and again create a forward shortcut. Thus $J_2 \in \mathcal{I}_2$. None of $x_1, \dots, x_k$ belongs to S_I, since otherwise the suffix beginning there would be shorter. The same closure argument gives $\mathrm{cl}_2(J_2) = \mathrm{cl}_2(I)$. Since $x_0 \in S_I$, one has $x_0 \notin \mathrm{cl}_2(I)$, and hence $J_2 + x_0 \in \mathcal{I}_2$. This is exactly $I \triangle V(P)$. A path consisting of a single vertex $x_0 \in S_I \cap T_I$ is immediate. □

3.4 The optimality certificate

The algorithm terminates because every augmentation increases $|I|$ by one. It remains to prove that stopping means optimality.

Completion of the proof of theorem 3.3. Let I be the common independent set at termination, and let R be the set of vertices reachable from S_I in $D(I)$. No vertex of T_I lies in R.

We first show that $I \cap R$ spans R in M_1. Consider $x \in R \setminus I$. Since $x \notin T_I$, the set $I + x$ contains a fundamental circuit C_1. For every $y \in C_1 - x$, the exchange $I - y + x$ is independent in M_1, so $x \to y$ is an arc. Reachability forces each such y to lie in R. Hence

$$C_1 \subseteq (I \cap R) + x,$$

and $x \in \mathrm{cl}_1(I \cap R)$. Therefore

$$r_1(R) = |I \cap R|.$$

Similarly, $I \setminus R$ spans $E \setminus R$ in M_2. If $x \in (E \setminus R) \setminus I$, then $x \notin S_I$ and $I + x$ has an M_2-fundamental circuit C_2. Every $y \in C_2 - x$ gives an arc $y \to x$. No such y can lie in R, or else x would be reachable. Thus $C_2 \subseteq (I \setminus R) + x$, and

$$r_2(E \setminus R) = |I \setminus R|.$$

Consequently,

$$r_1(R) + r_2(E \setminus R) = |I \cap R| + |I \setminus R| = |I|.$$

Together with the universal upper bound (1), this proves both the optimality of I and the min–max equality. □

Oracle complexity. With independence oracles, all arcs can be generated using $O(|E|^2)$ independence tests per augmentation. Since there are at most $r = \min\{r_1(E), r_2(E)\}$ augmentations, the basic implementation is polynomial. More sophisticated implementations avoid generating the entire exchange graph.

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