# HW1 47-853 Fall 2026

Note: the hints are on the second page. A first hint: if you find matroids too abstract, it's easier to think about graphic matroid (spanning trees) first :-) AI usage is NOT allowed.

**Problem 1** (Dual matroids) Recall that the dual matroid $M^* = (E, \mathcal{I}^*)$ of a matroid $M = (E, \mathcal{I})$ is defined as $B \in \mathcal{B}(M^*)$ if and only if $E \setminus B \in \mathcal{B}(M)$, where $\mathcal{B}(M)$ is the family of bases of matroid $M$.

1. Let $B$ be a basis, and $C$ be a circuit. Prove that for every $b \in B \cap C$, there exists $a \in C \setminus B$ such that $B - b + a$ is a basis.

2. A cocircuit of $M$ is defined as a circuit of the dual $M^*$. Alternatively, a cocircuit is a minimal *blocking set* of bases of $M$, i.e., it intersects all the bases of $M$ (think about why). Prove that for every circuit $C$ and cocircuit $C^*$, $|C \cap C^*| \neq 1$.

**Problem 2** (Multiple exchange property)

1. Let $r_M : 2^E \to \mathbb{Z}_{\geq 0}$ be the rank of matroid $M$. Derive the rank formula of the dual matroid $r_{M^*}$ in terms of $r_M$.

2. Prove the following exchange property of matroid bases: let $B, D$ be two disjoint bases of a matroid $M$. For every subset $X \subseteq B$, there exists a subset $Y \subseteq D$ such that both $B - X + Y$ and $D + X - Y$ are bases.

**Problem 3** (Shannon switching game) Consider the following Cut-Join game. Let $M = (E, \mathcal{I})$ be a matroid. The players Cut and Join alternately pick elements in $E$ that haven't been picked; Cut moves first. Join wins if the set picked by him spans the entire $E$; otherwise Cut wins.

1. Prove that if $E$ contains two disjoint bases, then Join has a winning strategy.

2. Prove that if $E$ does not contain two disjoint bases, then Cut has a winning strategy.

**Problem 4** (Derive matroids from polymatroids) Let $f : 2^E \to \mathbb{Z}_{\geq 0}$ be a polymatroid rank function, that is, it's monotone and submodular. Construct a matroid $M$ in the following way: duplicate every element $e \in E$ by $f(e)$ times; denote by $E'$ the duplicated ground set and by $P(e) \subseteq E'$ the collection of copies of $e$. A subset $I \subseteq E'$ is independent if and only if $\sum_{e \in F} |I \cap P(e)| \leq f(F)$ for every $F \subseteq E$; let $\mathcal{I}'$ be the collection of independent sets.

1. Prove that $M = (E', \mathcal{I}')$ defined this way is a matroid.

2. Construct another matroid $N = (E, \mathcal{I})$ in the following way: a subset $I \subseteq E$ is an independent set if and only if $|I \cap F| \leq f(F)$ for every $F \subseteq E$; let $\mathcal{I}$ be the collection of independent sets. Prove that $N$ is a matroid.

**Problem 5** (Transversal matroids and coverage function) Let $U$ be the universe and let $\{S_e\}_{e \in E}$ be a collection of subsets of $U$. Recall that a coverage function $f : 2^E \to \mathbb{Z}_{\geq 0}$ is defined as $f(F) := |\bigcup_{e \in F} S_e|$, which is the number of elements covered by sets in $F$. A *transversal matroid* $M = (E, \mathcal{I})$ is defined as follows: $I \subseteq E$ is independent if and only if there is an injective map $g : I \to U$ such that $g(e) \in S_e$ for every $e \in I$; let $\mathcal{I}$ be the collection of independent sets. Prove that $M$ is a matroid.

Hints:

- Problem 1-1. You may use the closure axiom.
- Problem 2-2. You may use the definition of dual matroid and Edmonds' formula for matroid intersection.
- Problem 3-1. You may use symmetric exchange property of matroids.
- Problem 3-2. You may use Edmonds' formula for base packing number, which is

  $$\min_{U \subseteq E,\ r(U) \neq r(E)} \left\lfloor \frac{|E \setminus U|}{r(E) - r(U)} \right\rfloor.$$

  The winning strategy of Cut is to always pick an element from $E \setminus U^*$ for a minimizer $U^*$. It is important that Cut moves first.
- Problem 4-2. You may want to recall the proof of matroid union.
- Problem 5. Use Problem 4.